On Some Properties of Class Q* $operators$

Abstract

The study of operators in Hilbert spaces holds significant importance, finding broad applications in diverse fields such as computer programming, financial mathematics and quantum physics. Many authors have extended the concept of normal operators in an attempt to provide practical solutions to complex problems in diverse fields. This paper focuses on a class Q^* operators in a Hilbert space H. An operator $T \in B(H)$ (where $B(H)$ represents bounded linear operators acting on H) is said to be class Q^* if $T^{*2}T^2 = (TT^*)^2$. By considering the properties of normal operators and other operators related to normal the study investigated the commutation relations and properties unique to class Q^* operators. The study shows that if two operators $T, S \in Q^*$ are such that the sum $(T+S)$ commutes with $(T+S)^*$, then $(T+S) \in Q^*$ and the product $TS \in Q^*$ if T and S commute with their adjoint. The results of this research are a valuable resource for mathematicians and physicists interested in the properties and applications of class Q^* operators fueling further innovations in functional analysis.

Keywords: Hilbert spaces, Normal operators, n-normal operators, adjoint, class Q* operators, Commutation relations.

1 Introduction

Throughout this paper $B(H)$ denotes the algebra of all bounded linear operators on Hilbert space H. A linear operator T on a Hilbert space H is said to be bounded if there exist a constant $c > 0$ such that $\|Tx\| \leq c\,\|x\|\, \forall x \in H$. T is called self-adjoint if $T = T^*$, invertible with inverse S if there exists $S \in B(H)$ such that $ST = I = TS$ where $I \in B(H)$ is the identity operator. An operator $T \in B(H)$ is said to be normal if it commutes with its adjoint i.e $(T^*T = TT^*)$, equivalently $T^*T - TT^* = 0$. An operator $T \in B(H)$ is said to be positive if $T^* = T$ and $\langle Tx, x\rangle \geq 0\ \forall x \in H$. An operator $T \in B(H)$ is said to be n-power normal if $T^nT^* = T^*T^n$ for $n \in \mathbb{N}$, class Q operator if for any $T \in Q$, $T^{*2}T^2 = (T^*T)^2$. $T \in B(H)$ is called a class Q^* if $T^{*2}T^2 = (TT^*)^2$ and Quasi-class Q if $T^{*3}T^3 - 2T^{*2}T^2 + T^*T \geq 0$. An operator $T \in B(H)$ is in class μ if $T^2 = -T^{*2}$ An operator $T \in B(H)$ is called an n-power-hyponormal operator if $T^nT^* \leq T^*T^n$. This class includes all normal, all n-normal and all hyponormal operators. An operator $T \in B(H)$ is Binormal if T^*T commutes with TT^*, That is $(T^*T)(TT^*) = (TT^*)(T^*T)$. [1] studied class Q^* operators and looked at properties such as adjoint, the inverse and operators unitarily equivalent to operators in class Q^* operators. This paper will study other properties of class Q^* operators that have not yet been studied.

2 Methodology

[5] introduced class Q operators on a Hilbert space where an operator $T \in B(H)$ is in class Q if $T^{*2}T^2 = (T^*T)^2$. Class Q operators were enlarged to n power class Q operators by [10]. [12] studied class Q^* operators, showed that class Q^* operators differ from class Q operators, and stated some of the properties as seen in Proposition 2.1.

Proposition 2.1: [12]
Let $T \in B(H)$, if $T \in Q^*$, then the following hold:

(i) T^{*2} is in Q^*

(ii) $T^{-1} \in Q^*$ provided it exists.

(iii) Any operator $S \in B(H)$ that is unitarily equivalent to T is also in Q^*.

Many authors have also investigated other properties such as Cartesian decomposition on different operators in Hilbert spaces. For instance, [1] studied Cartesian decomposition on n-normal operators and the results are stated in Proposition 2.2.

Proposition 2.2: [1]
Let $T \in B(H)$ with the Cartesian decomposition $T = A + iB$ where A and B are self-adjoint operators. Then T is a 2-normal operator if and only if B^2 commutes with A and A^2 commutes with B.

[10] proved that if $T \in B(H)$ with the Cartesian decomposition $T = A + iB$ where A and B are self adjoint operators, then T is binormal if and only if

(i) $AB^3 + B^3A = A^3B + BA^3$

(ii) $A^2BA + ABA^2 = B^2AB + BAB^2$

In [7], the author showed that if $T \in B(H)$ with the Cartesian decomposition $T = A + iB$, then $T \in \mu$ if and only if $A^2 = B^2$.

This paper will focus on various properties of class Q^* operators that have not yet been studied. These properties include convexity, Cartesian decomposition, sum and product, direct sum and the tensor product of class Q^* operators.

3 Results and discussions

Example 3.1 illustrates that class Q^* operators are not convex.

Example 3.1
Consider two operators $T, S \in Q^*$ such that $T = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$ and $S = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

Let $M = \frac{1}{2}T + \frac{1}{2}S$

$$M = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$M^* = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$M^{*2} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{9}{4} & -1 \\ 0 & \frac{1}{4} \end{bmatrix}$$

$$M^2 = \begin{bmatrix} \frac{3}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{3}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{9}{4} & 0 \\ -1 & \frac{1}{4} \end{bmatrix}$$

$$M^{*2}M^2 = \begin{bmatrix} \frac{9}{4} & -1 \\ 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \frac{9}{4} & 0 \\ -1 & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{97}{16} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

$$MM^* = \begin{bmatrix} \frac{3}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{9}{4} & -\frac{3}{4} \\ -\frac{3}{4} & \frac{1}{2} \end{bmatrix}$$

$$(MM^*)^2 = \begin{bmatrix} \frac{9}{4} & -\frac{3}{4} \\ -\frac{3}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{9}{4} & -\frac{3}{4} \\ -\frac{3}{4} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{90}{16} & -\frac{18}{16} \\ -\frac{33}{16} & \frac{13}{16} \end{bmatrix}$$

Now $M^{*2}M^2 \neq (MM^*)^2$, $M \notin Q^*$ and therefore Q^* is not convex.

Theorem 3.2 gives the results for the Cartesian decomposition on class Q^* operators on a Hilbert space.

Theorem 3.2

Let $T \in B(H)$ with the Cartesian decomposition $T = A + iB$ where A and B are self-adjoint operators. Then $T \in Q^*$ if

(i) $ABAB + BABA = A^2B^2 + B^2A^2$

(ii) $A^3B2B^3A2BA^3ABA^2 = 0$

Proof

Since $T \in Q^*$, then $T^{*2}T^2 = (TT^*)^2$
Now $T = A + iB$, $T^* = A - iB$

$$TT^* = (A + iB)(A - iB) = A^2 - iAB + iBA + B^2$$

$$T^{*2} = (A - iB)(A - iB) = A^2 - iAB - iBA - B^2$$
$$T^2 = (A + iB)(A + iB) = A^2 + iAB + iBA - B^2$$

$$\begin{aligned} T^{*2}T^2 &= (A^2 - iAB - iBA - B^2)(A^2 + iAB + iBA - B^2) \\ &= A^4 + iA^3B + iA^2BA - A^2B^2 - iABA^2 + ABAB \\ &\quad + AB^2A - iAB^3 - iBA^3 + BA^2B + BABA \\ &\quad + iBAB^2 - B^2A^2 - iB^2AB - iB^3A + B^4 \end{aligned} \tag{1}$$

$$
\begin{aligned}
(TT^*)^2 &= (A^2 - iAB + iBA + B^2)(A^2 - iAB + iBA + B^2) \\
&= A^4 - iA^3B + iA^2BA + A^2B^2 - iA^3B - ABAB \\
&\quad + AB^2A - iAB^3 + iBA^3 + BA^2B - BABA \\
&\quad + iBAB^2 + B^2A^2 - iB^2AB + iB^3A + B^4
\end{aligned} \tag{2}
$$

Since $T \in Q^*$, then equation (1) must equate to equation (2).
On further simplification,

$$
\begin{aligned}
T^{*2}T^2 &= (TT^*)^2 \\
&= iA^3B - A^2B^2 - iABA^2 + ABAB - iAB^3 + BA^2B + BABA - B^2A^2 - iB^3A \\
&= -iA^3B + A^2B^2 - iA^3B - ABAB + iBA^3 - BABA + B^2A^2 + iB^3A
\end{aligned}
$$

Equating the real part, we have

$$ABAB + BABA - A^2B^2 = A^2B^2 + B^2A^2 - ABAB - BABA$$

$$\implies ABAB + BABA = A^2B^2 + B^2A^2$$

$$\implies AB = BA$$

Thus A commutes with B

Equating the imaginary part, we have

$$iA^3B - iABA^2 - iBA^3 - iB^3A = -iA^3B - iA^3B + iBA^3 + iB^3A$$

$$A^3B - ABA^2 - BA^3 - B^3A = -A^3B - A^3B + BA^3 + B^3A$$

Which means

$$2A^3B - 2B^3A - 2BA^3 - ABA^2 = 0$$

□

Theorem 3.3 states the result on the commutation relation in class Q^* operators.

Theorem 3.3
Let $T$and S be bounded linear operators in Q^* such that the sum $(T+S)$ commutes with $(T+S)^*$. Then $(T+S) \in Q^*$.

Proof

$$
\begin{aligned}
(T+S)^{*2}(T+S)^2 &= (T+S)^*(T+S)^*(T+S)(T+S) \\
&= (T+S)^*(T+S)(T+S)^*(T+S) \\
&= (T+S)(T+S)^*(T+S)(T+S)^* \\
&= (T+S)(T+S)(T+S)^*(T+S)^* \\
&= (T+S)^2(T+S)^{*2} \\
&= ((T+S)(T+S)^*)^2
\end{aligned}
$$

Hence $(T+S) \in Q^*$

□

Example 3.4
Let T and S be operators in class Q^* such that

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } S = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(T+S) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$(T+S)^* = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$(T+S)(T+S)^* = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$$

$$(T+S)^*(T+S) = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$$

Now

$$(T+S)^{*2} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$$

$$(T+S)^2 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$$

$$(T+S)^{*2}(T+S)^2 = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 81 & 0 \\ 0 & 16 \end{bmatrix}$$

$$(TT^*)^2 = (\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix})^2 = (\begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix})^2 = \begin{bmatrix} 81 & 0 \\ 0 & 16 \end{bmatrix}$$

Hence $(T+S) \in Q^*$

Remark 3.5
If two operators $T, S \in Q^*$ are such that the sum $T+S$ does not commute with $(T+S)^*$ then $T+S$ is not necessarily in class Q^* operators.
Example 3.6 shows that the sum of two operators T and S in class Q^* such that $(T+S)(T+S)^* \neq (T+S)^*(T+S)$ does not belong to class Q^*.

Example 3.6

Let $T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $S = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ be two operators in class Q^*.

$$T+S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$$

$$(T+S)^* = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$(T+S)(T+S)^* = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 1 \\ 1 & 5 \end{bmatrix}$$

$$(T+S)^*(T+S) = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 10 & -1 \\ -1 & 5 \end{bmatrix}$$

Therefore $(T+S)$ is not normal.
Consequently,

$$(T+S)^2 = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & -5 \\ 5 & 3 \end{bmatrix}$$

$$(T+S)^{*2} = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$(T+S)^{*2}(T+S)^2 = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 8 & -5 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 89 & -25 \\ -25 & 34 \end{bmatrix}$$

$$((T+S)(T+S)^*)^2 = (\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix})^2 = \begin{bmatrix} 10 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 10 & 1 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 102 & 15 \\ 15 & 6 \end{bmatrix}$$

Clearly $(T+S)^{*2}(T+S)^2 \neq ((T+S)(T+S)^*)^2$ and so$(T+S) \notin Q^*$

The result on the product of two class Q^* operators is given in theorem 3.7

Theorem 3.7
Let $T$and S be bounded linear operators in Q^* such that T and S commute with their adjoint. Then $(TS) \in Q^*$.

Proof

$$\begin{aligned}
(TS)^{*2}(TS)^2 &= (TS)^*(TS)^*(TS)(TS) \\
&= S^*T^*S^*T^*TSTS \\
&= S^*T^*S^*TT^*STS \\
&= T^*S^*TS^*ST^*TS \\
&= T^*TS^*SS^*TST^* \\
&= TT^*SS^*TS^*ST^* \\
&= TST^*TS^*SS^*T^* \\
&= TSTT^*SS^*T^*S^* \\
&= TSTST^*S^*T^*S^* \\
&= (TS)^2((T^*S^*))^2 \\
&= (TS)^2((TS)^*))^2 \\
&= ((TS)(TS)^*)^2
\end{aligned}$$

Now $(TS)^{*2}(TS)^2 = ((TS)(TS)^*)^2$ implying that $(TS) \in Q^*$

□

Example 3.8

Consider two operators $T, S \in Q^*$ such that $T = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$ and $S = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$T^* = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$ and $S^* = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$$TS^* = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} -2i & -2i \\ 2i & -2i \end{bmatrix}$$

$$S^*T = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} -2i & -2i \\ 2i & -2i \end{bmatrix}$$

Therefore $TS^* = S^*T$

$$T^*S = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 2i & -2i \\ 2i & 2i \end{bmatrix}$$

$$ST^* = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 2i & -2i \\ 2i & 2i \end{bmatrix}$$

Therefore $T^*S = ST^*$

$TS = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} -2i & -2i \\ 2i & -2i \end{bmatrix}$

Let the product TS be M $M$is a class Q^* operator since

$M = \begin{bmatrix} -2i & -2i \\ 2i & -2i \end{bmatrix}, M^* = \begin{bmatrix} 2i & -2i \\ 2i & 2i \end{bmatrix}$

$$M^{*2} = \begin{bmatrix} 2i & -2i \\ 2i & 2i \end{bmatrix} \begin{bmatrix} 2i & -2i \\ 2i & 2i \end{bmatrix} = \begin{bmatrix} 0 & 8 \\ -8 & 0 \end{bmatrix}$$

$$M^2 = \begin{bmatrix} -2i & -2i \\ 2i & -2i \end{bmatrix} \begin{bmatrix} -2i & -2i \\ 2i & -2i \end{bmatrix} = \begin{bmatrix} 0 & -8 \\ 8 & 0 \end{bmatrix}$$

$$M^{*2}M^2 = \begin{bmatrix} 0 & 8 \\ -8 & 0 \end{bmatrix} \begin{bmatrix} 0 & -8 \\ 8 & 0 \end{bmatrix} = \begin{bmatrix} 64 & 0 \\ 0 & 64 \end{bmatrix}$$

$$MM^* = \begin{bmatrix} -2i & -2i \\ 2i & -2i \end{bmatrix} \begin{bmatrix} 2i & -2i \\ 2i & 2i \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$

$$(MM^*)^2 = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 64 & 0 \\ 0 & 64 \end{bmatrix}$$

Therefore M is class Q^* since $M^{*2}M^2 = (MM^*)^2$.

Remarks 3.9
The commutation relation of the operator with the adjoint of the other should not be ignored; otherwise, the operators' product will not be class Q^* as seen in example 3.10.

Example 3.10

Let $T = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ be two operators in class Q^*.

$$T^* = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, S^* = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

$$TS^* = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix} \neq S^*T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$$

$$T^*S = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \neq ST^* = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$$

Consequently, the product $TS \notin Q^*$.

Let $M = TS = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$

$$M^* = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$$

$$M^{*2} = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$M^2 = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$M^{*2}M^2 = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$MM^* = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$(MM^*)^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}$$

Hence$M^{*2}M^2 \neq (MM^*)^2$.

The study of direct sum and tensor product of operators in Hilbert spaces has been a topic of interest for many researchers. [5] proved that the direct sum and tensor product of two operators in SN are in SN. Later, [7] showed that the direct sum and tensor product of two operators in μ are in μ. [8] showed that if $T_1, T_2, ...T_m$ are n-power-hypornormal operators in $B(H)$, then $T_1 \oplus T_2 \oplus ... \oplus T_m$ and $T_1 \otimes T_2 \otimes ... \otimes T_m$ are n-power-hyponormal operators. Theorem 3.11 gives the results of the direct sum and tensor product of operators in class Q^*.

Theorem 3.11
Let $T_1, T_2, ...T_m$ be normal operators in class Q^*, Then;

(i) $T_1 \oplus T_2 \oplus ... \oplus T_m \in Q^*$

(ii) $T_1 \otimes T_2 \otimes ... \otimes T_m \in Q^*$

Proof. (i)
$(T_1 \oplus T_2 \oplus ... \oplus T_m)^{*2}(T_1 \oplus T_2 \oplus ... \oplus T_m)^2$
$= (T_1 \oplus T_2 \oplus ... \oplus T_m)^*(T_1 \oplus T_2 \oplus ... \oplus T_m)^*(T_1 \oplus T_2 \oplus ... \oplus T_m)(T_1 \oplus T_2 \oplus ... \oplus T_m)$
$= (T_1^* \oplus T_2^* \oplus ... \oplus T_m^*)(T_1^* \oplus T_2^* \oplus ... \oplus T_m^*)(T_1 \oplus T_2 \oplus ... \oplus T_m)(T_1 \oplus T_2 \oplus ... \oplus T_m)$
$= (T_1^* \oplus T_2^* \oplus ... \oplus T_m^*)(T_1^*T_1 \oplus T_2^*T_2 \oplus ... \oplus T_m^*T_m)(T_1 \oplus T_2 \oplus ... \oplus T_m)$
$= (T_1^* \oplus T_2^* \oplus ... \oplus T_m^*)(T_1T_1^* \oplus T_2T_2^* \oplus ... \oplus T_mT_m^*)(T_1 \oplus T_2 \oplus ... \oplus T_m)$
$= (T_1^* \oplus T_2^* \oplus ... \oplus T_m^*)(T_1 \oplus T_2 \oplus ... \oplus T_m)(T_1^* \oplus T_2^* \oplus ... \oplus T_m^*)(T_1 \oplus T_2 \oplus ... \oplus T_m)$
$= (T_1^*T_1 \oplus T_2^*T_2 \oplus ... \oplus T_m^*T_m)(T_1^*T_1 \oplus T_2^*T_2 \oplus ... \oplus T_m^*T_m)$
$= (T_1T_1^* \oplus T_2T_2^* \oplus ... \oplus T_mT_m^*)(T_1T_1^* \oplus T_2T_2^* \oplus ... \oplus T_mT_m^*)$
$= (T_1 \oplus T_2 \oplus ... \oplus T_m)(T_1^* \oplus T_2^* \oplus ... \oplus T_m^*)(T_1 \oplus T_2 \oplus ... \oplus T_m)(T_1^* \oplus T_2^* \oplus ... \oplus T_m^*)$
$= (T_1 \oplus T_2 \oplus ... \oplus T_m)(T_1^*T_1 \oplus T_2^*T_2 \oplus ... \oplus T_m^*T_m)(T_1^* \oplus T_2^* \oplus ... \oplus T_m^*)$
$= (T_1 \oplus T_2 \oplus ... \oplus T_m)(T_1T_1^* \oplus T_2T_2^* \oplus ... \oplus T_mT_m^*)(T_1^* \oplus T_2^* \oplus ... \oplus T_m^*)$
$= (T_1 \oplus T_2 \oplus ... \oplus T_m)(T_1 \oplus T_2 \oplus ... \oplus T_m)(T_1^* \oplus T_2^* \oplus ... \oplus T_m^*)(T_1^* \oplus T_2^* \oplus ... \oplus T_m^*)$
$= (T_1 \oplus T_2 \oplus ... \oplus T_m)^2(T_1^* \oplus T_2^* \oplus ... \oplus T_m^*)^2$
$= ((T_1 \oplus T_2 \oplus ... \oplus T_m)(T_1^* \oplus T_2^* \oplus ... \oplus T_m^*))^2$
Now, $(T_1 \oplus T_2 \oplus ... \oplus T_m)^{*2}(T_1 \oplus T_2 \oplus ... \oplus T_m)^2 = ((T_1 \oplus T_2 \oplus ... \oplus T_m)(T_1^* \oplus T_2^* \oplus ... \oplus T_m^*))^2$
Hence $T_1 \oplus T_2 \oplus ... \oplus T_m \in Q^*$

Proof. (ii)
Let $x_1, x_2, ...x_m \in H$
$(T_1 \otimes T_2 \otimes ... \otimes T_m)^{*2}(T_1 \otimes T_2 \otimes ... \otimes T_m)^2(x_1 \otimes x_2 \otimes ... \otimes x_m)$
$= (T_1 \otimes T_2 \otimes ... \otimes T_m)^*(T_1 \otimes T_2 \otimes ... \otimes T_m)^*(T_1 \otimes T_2 \otimes ... \otimes T_m)(T_1 \otimes T_2 \otimes ... \otimes T_m)(x_1 \otimes x_2 \otimes ... \otimes x_m)$
$= (T_1^* \otimes T_2^* \otimes ... \otimes T_m^*)(T_1^* \otimes T_2^* \otimes ... \otimes T_m^*)(T_1 \otimes T_2 \otimes ... \otimes T_m)(T_1 \otimes T_2 \otimes ... \otimes T_m)(x_1 \otimes x_2 \otimes ... \otimes x_m)$
$= (T_1^* \otimes T_2^* \otimes ... \otimes T_m^*)(T_1^*T_1 \otimes T_2^*T_2 \otimes ... \otimes T_m^*T_m)(T_1 \otimes T_2 \otimes ... \otimes T_m)(x_1 \otimes x_2 \otimes ... \otimes x_m)$
$= (T_1^* \otimes T_2^* \otimes ... \otimes T_m^*)(T_1T_1^* \otimes T_2T_2^* \otimes ... \otimes T_mT_m^*)(T_1 \otimes T_2 \otimes ... \otimes T_m)(x_1 \otimes x_2 \otimes ... \otimes x_m)$
$= (T_1^* \otimes T_2^* \otimes ... \otimes T_m^*)(T_1 \otimes T_2 \otimes ... \otimes T_m)(T_1^* \otimes T_2^* \otimes ... \otimes T_m^*)(T_1 \otimes T_2 \otimes ... \otimes T_m)(x_1 \otimes x_2 \otimes ... \otimes x_m)$
$= (T_1^*T_1 \otimes T_2^*T_2 \otimes ... \otimes T_m^*T_m)(T_1^*T_1 \otimes T_2^*T_2 \otimes ... \otimes T_m^*T_m)(x_1 \otimes x_2 \otimes ... \otimes x_m)$
$= (T_1T_1^* \otimes T_2T_2^* \otimes ... \otimes T_mT_m^*)(T_1T_1^* \otimes T_2T_2^* \otimes ... \otimes T_mT_m^*)(x_1 \otimes x_2 \otimes ... \otimes x_m)$
$= (T_1 \otimes T_2 \otimes ... \otimes T_m)(T_1^* \otimes T_2^* \otimes ... \otimes T_m^*)(T_1 \otimes T_2 \otimes ... \otimes T_m)(T_1^* \otimes T_2^* \otimes ... \otimes T_m^*)(x_1 \otimes x_2 \otimes ... \otimes x_m)$
$= (T_1 \otimes T_2 \otimes ... \otimes T_m)(T_1^*T_1 \otimes T_2^*T_2 \otimes ... \otimes T_m^*T_m)T_1^* \otimes T_2^* \otimes ... \otimes T_m^*)(x_1 \otimes x_2 \otimes ... \otimes x_m)$
$= (T_1 \otimes T_2 \otimes ... \otimes T_m)(T_1T_1^* \otimes T_2T_2^* \otimes ... \otimes T_mT_m^*)T_1^* \otimes T_2^* \otimes ... \otimes T_m^*)(x_1 \otimes x_2 \otimes ... \otimes x_m)$
$= (T_1 \otimes T_2 \otimes ... \otimes T_m)(T_1 \otimes T_2 \otimes ... \otimes T_m)(T_1^* \otimes T_2^* \otimes ... \otimes T_m^*)(T_1^* \otimes T_2^* \otimes ... \otimes T_m^*)(x_1 \otimes x_2 \otimes ... \otimes x_m)$
$= (T_1 \otimes T_2 \otimes ... \otimes T_m)^2(T_1^* \otimes T_2^* \otimes ... \otimes T_m^*)^2(x_1 \otimes x_2 \otimes ... \otimes x_m)$
$= ((T_1 \otimes T_2 \otimes ... \otimes T_m)(T_1^* \otimes T_2^* \otimes ... \otimes T_m^*))^2(x_1 \otimes x_2 \otimes ... \otimes x_m)$
Therefore $T_1 \otimes T_2 \otimes ... \otimes T_m \in Q^*$

4 Conclusion

This research focused on investigating class Q^* operators and their properties in Hilbert spaces. Several properties, including convexity, cartesian decomposition, sum and product and direct sum and tensor product have been investigated. Future research plans to explore the spectral properties of class Q^* of operators.

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