A SYMBOLIC SOLUTION APPROACH FOR CHAOTIC INDUCED PROBLEMS

ABSTRACT

This work investigates dynamical chaos of systems resulting from bifurcation of non-linear system. Dynamical behaviours of some complex nonlinear systems are considered. The work explores chaotic situations exhibiting irregular random behaviors due to the system ergodic properties. Therefore, a symbolic solution approach has been suggested. This approach includes a feedback control mechanism, discretization of the nonlinear coupled equation and the transformation of the nonlinear system into a linear periodic system. The solution starts with normalizing the dynamical control system using state space variables of the system dynamics and then linearized. The resulting system is transformed into special polynomials. The principles and concepts of chaos theory are relayed in the work. The results showed that the linearized system is asymptotically stable. An illustration is provided.

KEYWORDS: Chaos, Symbolic Approach, Normalization, Linear Periodic System, Picard's iterations, shifted Chebyshev Polynomials.

1. Introduction

A Chaotic system is a dynamic system and a dynamical system is a system that changes over time [1]. Linear systems usually respond between cause and effect while nonlinear systems react disproportionately to Chaos([2]). Thus, we can describe chaos theory as the study of systems that are deterministic[3], Some definitions of Chaos can be regarded as a low-dimensional dynamical system with complicated non periodic attracting orbits. While, some describe chaos as a special type of order without periodicity or having random evolution (Faghani *et al*, [4]; Fradkov and Chen, [5]). chaos has previously been described as a dynamical system showing likely dependence on initial conditions with respect to a closed invariant set having more than one orbits [6] and later with different perturbation types[7] . That is, they `are identified by their reliance on initial conditions.

However, Tabor[8] opines that a chaotic solution to a deterministic equation is a solution whose outcome displays small changes in initial conditions leading to great differences in outcome and whose evolution appears to be random. Thus, Margielewicz *et al* [9] focused on nonlinear dynamics of flexible coupling model. Moreover, Rasband[10] defined chaos as an observations of a system through measurement that vary unpredictably. Hence, we say that observations are chaotic if there are no regularity or order. Some early work on chaos and its dynamics described what happens within the system. Lototsky and Rozovskii [11] described a generalized solution method for stochastic evolution equation. In summary then, Chaos theory is a theory of randomness of complex systems, their underlying patterns, feedback loop, repetition, self similarity, fractals and reliance on initial conditions[12]. When the systems are deterministic, long-term prediction of their behavior becomes impossible. This implies that their future behavior has a deterministic nature that makes the observed system to be predictable. This idea encountered in some works from literature have opened up new dimensions to the study of chaos. Karimi and Nik[13] considered this application by presenting an algorithm for a piecewise spectral homotopy for optimal and adaptive control problem of hyperchaotic system. Also, a sensitivity analyses[14] resulting in these dynamics have been carried out by some researchers. Wei *et al*[15] looked at the uncertainty based optimization for ships by conducting a sensitivity analysis of polynomial chaos expansion having no or little computational cost. This was an improved collocation method suggested by them to reduce sample points

Let us consider the work of Morgan[16] who addressed the confusion behind the view of chaos theory as a metaphor as well as its under use application for economic research. This leads to the transition of the system from one state to another and can be represented by diagrams of the bifurcation paths. This has been illustrated in this work. Chaos generates the Concepts of nonlinearity [17], with the belief that complex systems cannot be controlled, but can be accessed and even influenced through the myriad of feedback closed loops[18]. This notion had been fully explained by Lorenz's [19] metaphor of the butterfly. However, the work of Zamani *et al* [20] engaged in the synthesis of the control systems which can be reduced to a fixed-point computation over finite-state abstraction based on some dynamical properties. They noted that their technique imposes no restrictions on the sampling time. The Floquet Transition Multiplier(FTM) is a very important step in stability analysis and obtained by evaluating the Symbolic Transient Multiplier (STM) with the principal period T. Sinha and Butcher[21] then described this process as a symbolic computation technique for obtaining the linear periodic system explicitly as a function of the system parameters. This method involves the use of Picard's iterations with its expansion into shifted Chebyshev polynomials. The stability analysis uses the FTM and the concept of transition matrix. From existing literature, this has been shown to be very efficient[9] and applied in different areas such as finance[22], special polynomials[23], error analysis[24], Fuzzy control[25] and stability analysis([26]) .

Next, we present a full-state feedback control system, transform the parameters into linear quantities through symbolic approach and then normalize the resulting system. We introduce the sequence in section 2 and show that the approach converges and that it also solves the system[27] in sections 3. The bifurcation paths is presented in figure1.

2. Linear feedback control through the Symbolic Approach

Given the nonlinear dynamic control system with time-periodic coefficients denoted by:

$$y = F(x,u(x),\alpha,t) \quad (1)$$

Where $x \in R^n, u \in R, \alpha \in R^m$ and

$$F(x,u,\alpha,t+T) = F(x,u,\alpha,t). \quad (2)$$

The vector $\alpha \in /^m$ contains the different parameters of the system. $\{x=0, u=0\}$ is assumed to be the equilibrium point fixed as a reference point. To control the state input, $u(x)$ in (1), it must be a linear function of x and can be denoted as

$$\dot{y} = \dot{F}(x,\alpha,t) + L_u(\alpha,t)u \quad (3)$$

In addition, L_u is a matrix with respect to the input variable. Assume that $u(x) \in R$ such that for any $\alpha \in \{\alpha_j, \alpha_i\}$, we have a desired periodic orbit $x_t = x_t(t+iT)$ where $i = 1,2,3,...$ or $1/2, 1/3, 1/4,...$

Moreover, the periodicity of this orbit and the system are integer multiples of each other or in converse, a closed-loop system with a quasi-periodic coefficients that can be determined by applying a law of control equation comprising of two parts: forward-feed u_f and a feed-back u_b given as

$$L_u(t)u = L_u(u_f + u_b) = y - F(x_t, \alpha, t) - L_u k(x - x_t),\ k = \{k_1, k_2, .. k_n\} \quad (4)$$

equation (4) is the result of the first linearization of the systems parameters. To normalize the periods, we define $\dot{x} = x - x_t$, as the error resulting from the actual and desired trajectories[24]. Equation (3) in terms of the error variables becomes

$$\dot{x} = F(x, \alpha, t) - L_u(\alpha, t)kx. \quad (5)$$

and

$$\dot{x} = (L_X(\alpha, t) - L_u(\alpha, t)x := L_k(\alpha, t)x \quad (6)$$

as linearization at the zero-equilibrium point, with L_k as a time-periodic matrix with period T.

2.1 ASSUMPTIONS

i)Equation (6) is the linearized system and $\boldsymbol{k}$ is the control vector, such that $\lim_{t\to\infty} x = 0$.
ii) the eigenvalues of $\varphi_k(\alpha, k)$ are within the unit circle of the complex plane,
iii) let the Routh- Hurwitz criterion be satisfied,
iv)the stability of equation (6) is engaged by its FTM.
Then

Proposition 2.1

The linearized system is asymptotically stable according to the Floquet theory

Proof: Express the linearized system, that is equation (6) in the integral form as

$$x(t) = x(0) + \int_0^t L_k(\alpha, \tau)x(\tau)d\tau \quad (7)$$

use the Picard iteration method to obtain the $(k+1)$th approximation given as

$$x^{(k+1)}(t) = x(0) + \int_0^t L_k(\alpha, \tau_k)x^{(k)}(\tau_k)d\tau_k$$

$$= \mathrm{I} + [\int_0^t L_k(\alpha, \tau_k)d\tau_k + \int_0^t L_k(\alpha, \tau_{k-1})d\tau_{k-1} + \int_0^t L_k(\alpha, \tau_{k-2})d\tau_{k-2} + \ldots + \int_0^{\tau_1} L_k(\alpha, \tau_0)d\tau_0 \ldots d\tau_k]x(0), \quad (8)$$

where $\tau_0, \ldots \tau_k$ are dummy variables. The expression in the square brackets indicate the approximation to the solution matrix $\Phi(t, \alpha) = \begin{bmatrix} \boldsymbol{\Phi}_{k_1} \\ \boldsymbol{\Phi}_{k_2} \\ \boldsymbol{\Phi}_{k_3} \\ \vdots \\ \boldsymbol{\Phi}_{k_k} \end{bmatrix}$ of $L_k(\alpha, \tau)$since it is obtained after a finite number of iterations. Transforming the periods from $\boldsymbol{t}$ ***to*** $\boldsymbol{T\tau}$, the one-periodic system matrix $\bar{L}_k(\alpha, \tau)$ is expanded via m-shifted Chebychev polynomials of the first kind. Considering the product matrices associated with Chebychev polynomials, equation (7) yields a polynomial expression for $\Phi_k(\alpha,k)$ in terms of the system parameters α and the unknown control gains k. Then, the eigenvalues $\boldsymbol{\Phi_k(\alpha,t)} = \Phi_k(\alpha,k)$ can be positioned at the desired locations by choosing appropriate values of k for any $\alpha \in \{\alpha_{j,}\alpha_i\}$. Φ_k contains higher degree polynomial expressions of k and α. k may be selected by applying a stability criterion for the maps Φ_k (the Routh-Hurwitz criterion). The value of k so selected guarantees asymptotic stability of equation (7) from its entire parameters range $\{\alpha_j,\alpha_i\}$.□

2.2 **Routh-Hurwitz Criterion** is a necessary and sufficient condition for the stability of a linear system. It is a method of determining the stability behaviour of a dynamical system. It confirms whether or not the dynamic system frequency equation has roots containing positive real part which will make the system unstable.

2.2.1. **Floquet Theory** deals with the resulting ODEs involving the class of solutions to the periodic linear differential equation(7) having piecewise continuous periodic function and defines the stability state of the solution.

3. Feedback linearization and Local Stability

It is important to find a set of simple coordinate transformations that can convert the original nonlinear time-periodic control problem to a dynamically equivalent linear time-periodic system. Let us consider the nonlinear periodic system of the full-state feedback control using Symbolic Approach

$$\dot{x} = F(x, u(x), \alpha, t) \tag{9}$$

at some fixed value of the system parameters $\alpha_1 \in (\alpha_j, \alpha_h)$. Also, nonlinear control equation may provide an increased region of stability when compared with linearization methods only. The equation (9) is parameter independent as

$$\dot{x} = F(x,t) + L_U(t)u \tag{10}$$

Applying positive forward and a negative backward feedback control in the form of

$$\boldsymbol{L_u(t) = L_u(t)(u_v + u_n) = x - F(x_\tau, t) + L_n(t)u = L_u(t)} \tag{11}$$

The closed-loop system in the error variable is obtained by substituting equation (10) into equation (11). Also, see [26] on how to go about the expansion. Expanding the closed-loop system and applying the Taylors series in the state variables, gives

$$\overset{*}{y} = L_y(t)y + Q_y(y,t) + C_y(y,t) + \ldots + F_r(y,t) + L_u(t)u_n \tag{12}$$

Where Q_y, C_y *and* F_r are symmetric quadratic, cubic and rth-order functions of y with T-periodic coefficients, respectively. Next, Lyapunov-Floquet transformation(27) of $y = Q(t)\bar{y}$ is applied to convert the constant linear matrix into a canonical form as

$$\dot{y} = k\bar{y} + \bar{Q}_y(\bar{y},t) + \bar{C}(\bar{y},t) + \ldots + \bar{F}_r(\bar{y},t) + \bar{L}_u(t)u_n \quad (13)$$

4. Numerical Examples.

Consider the nonlinear coupled system

$$\frac{d}{dt}x_1(t) = x_2(t)^2 - 4\,,\ \frac{d}{dt}x_2(t) = x_1(t) - 1 + u(t)\,,\ y(t) = x_1(t) + x_2(t)\,, \quad (14)$$

with initial conditions

$$u(t) = 0, x_1(t) = 2, x_2(t) = 4 \text{ , and } constraints = [0 < x_1(t)] \text{ .}$$

Using Symbolic Approach, the bifurcation path and the stability of the system is obtained below;

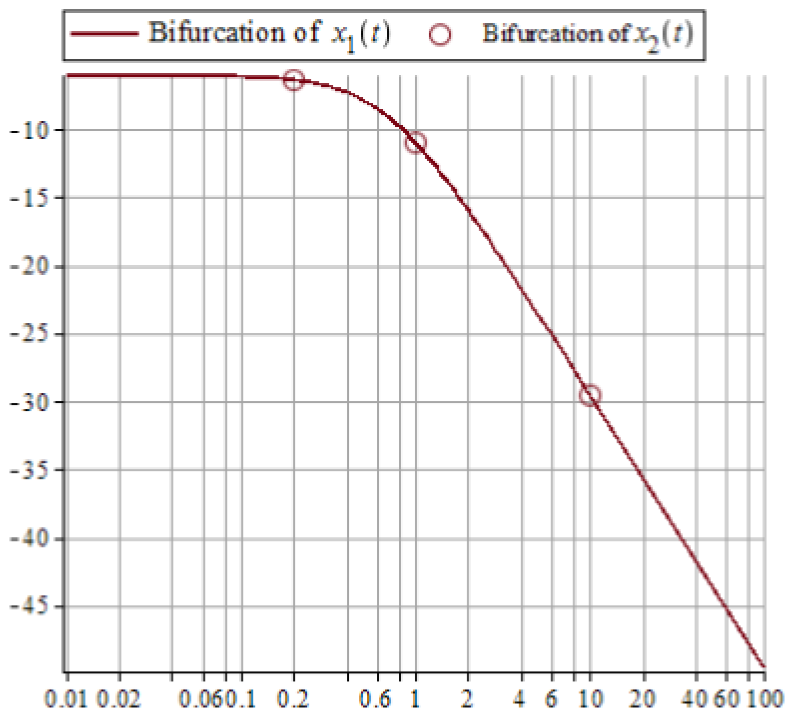

Figure1: Bifurcation of the coupled nonlinear dynamical system through the Symbolic Approach

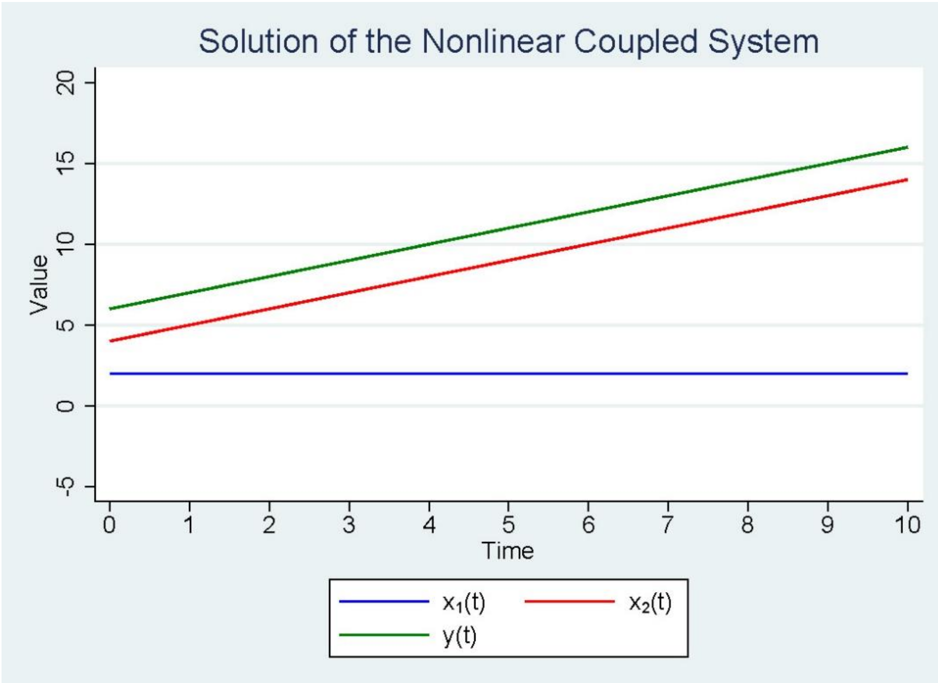

Figure2: zero equilibrium point of the nonlinear system

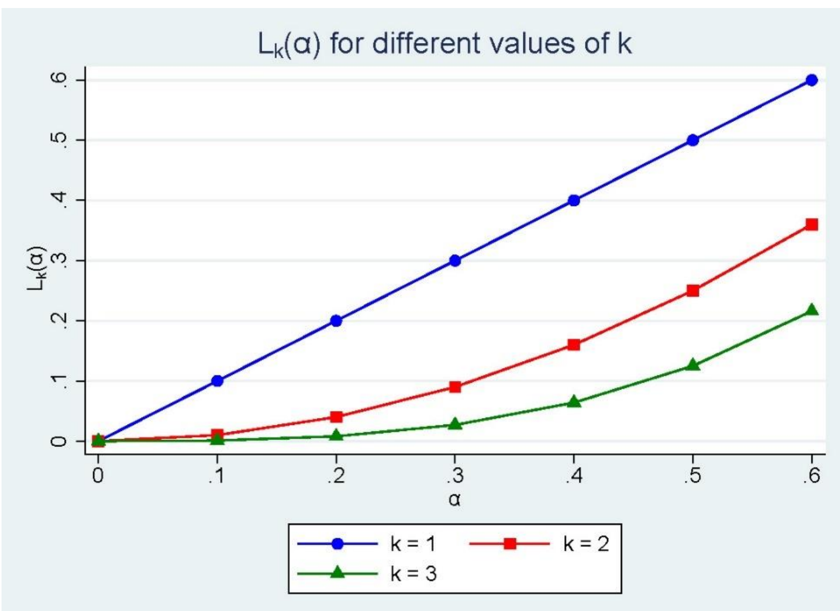

Figure3: the input values for k=1,2,3

5 CONCLUSION

A periodic system is stable if its characteristic multiplier belongs to the unit open disc on the complex plane or if the characteristic exponent is negative. In control system theory, the Routh–Hurwitz stability criterion is a necessary and sufficient condition for the stability of a linear time invariant control system. The major focus is the transformation of the nonlinear system into a linear and normalized one. Considering the chaotic system, the process is reformulated as a nonlinear equation modeled as a coupled dynamical system. Attempts were made to analyze the chaotic system to show stability of the periodic solution. The modeled equation is transformed into Lyapunov equation from where the FTM is evaluated. The eigen-values (characteristic multipliers) of the periodic linear system is extracted and solution obtained. The concepts from the theory of chaos is useful in our everyday economic bargaining. Figure1 shows the path of bifurcation

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