Characterization of soft sets as soft semigroups

Abstract. Several algebraic structures of soft sets have existed in the literature, with many of their applications in problems that involve decision making. However, some semigroup properties associated with soft sets have not been exhausted. Hence, in this paper, we present a characterization of soft sets as an algebraic structure of semigroups, then present also some semigroup properties affiliated to it.

Keywords: Soft Sets, Soft Semigroups, Soft Morphisms, Soft Regular Semigroups.

Mathematics Subject Classification: 20M10

1. Introduction

It is familiar knowledge that for tackling some real world problems, mathematical models have turned out to be a desired tool. The problems cut across several fields including engineering, social sciences, natural sciences, medicine, economics, computer science etc.

Due to the unpredictability associated with real world situations, classical methods in mathematics may not be the appropriate tool for tackling them. To this effect, Zadeh [19] initiated the exceptional theory of fuzzy sets to tackle certain unpredictability her termed "fuzziness", which has to do with the partial membership of an element in a given set.

Despite Zadeh's fuzzy set proving to be fruitful in managing unpredictability and partial membership of an element in a set, it falls short in terms of its inability to manage all sort of unpredictability existing in different real physical situations. This led to a search for a more efficacious theory.

Molodstov [10] soft set theory provided the needed succor to overcoming these perculiarities. His theory adopted a completely divergent approach in portraying unpredictability. It involves the use of parameterization to manage unpredictability related to real world situations. Soft set theory distinguishes itself from probability and fuzzy theories in the sense that it does not uphold a precise quantity. Molodstov applied his theory to various concepts, such as, smoothness of functions, measurement theory and so on.

A soft set classifies elements with reference to a collection (set) of parameters. It has been established that a soft set is more extensive in nature and has more capacity to manage information with unpredictability. It has also been established that fuzzy set and rough set are special kinds of soft sets. The fundamental theory of soft sets has in recent years attracted the interest and curiosity of so many researchers, leading to extensive studies in the area with ground breaking discoveries and results churned out time and again more research is still going on in this field with various applications in the field of science and technology.

Maji and Roy [8] were first to provide a pragmatic application of soft sets in decision making issues as well as provide definitions of some soft binary operations. These includes the operations AND, OR, UNION and INTERECTION of two soft sets.

Apart from the rich application of soft sets, its algebraic structures have been studied extensively. Aktaz and Cagman [1] studied soft groups and their properties. The notion of soft semirings was initiated by Feng et al [6], while Acar et al [3] considered the concept of soft rings and soft ideals of soft rings. Jayanta [7] expounded the study of the algebraic structure of the set of all soft sets defined on a fixed universe and showed that on a fixed set of parameters, this set is a Boolean algebra. A new approach to the study of these structures was initiated by Mohammed et al [11].

Further researchers like Maji et al [8], Alkhazaleh et al [4], Pinak [12], Singh et al [15] have combined soft sets with other sets including fuzzy sets, rough sets and multi sets, to generate hybrid structures like fuzzy soft sets, rough soft sets, multi soft sets amongst others.

The application of these aforementioned structures in decision making, medical diagnosis, forecasting, etc have been studied by the likes of Roy and Maji [13] , Das and Borgohaim [5], Rajarajeswan and Dhame [14], Udhaya et al [17], Sai [16], and so on.

Several works existing in Literature have been carried out to establish the algebraic structure of soft sets with applications in problems involving decision making. To the best of our knowledge, however, some semigroup properties affiliated to soft sets have not been completely exhausted.

Therefore, this work characterizes soft sets as an abstract structure of semigroups and presents some semigroup properties affiliated to it.

2. Preliminaries

We now recall some definitions and some established results which will be important in this paper. The reader is to consult [2], [7],[8], [9] and [18] for notations and terminologies not captured in this paper.

Definition 2.1 [10]. Let $\mathcal{U}$ and $\mathcal{E}$ denote the universe and set of parameters that describe the elements in $\mathcal{U}$, respectively (elements in $\mathcal{E}$ can also be called attributes of the elements in $\mathcal{U}$).

Suppose $\mathcal{A} \subseteq \mathcal{E}$ and $\mathcal{P}(\mathcal{U})$ is the set of subsets of $\mathcal{U}$. Then the pair $(\mathcal{F}, \mathcal{A})$ is called the soft set of $\mathcal{A}$, where $\mathcal{F}$ is a mapping such that $\mathcal{F} : \mathcal{A} \to \mathcal{P}(\mathcal{U})$.

$(\mathcal{F}, \mathcal{A})$ can be considered as a parameterized collection of subsets of $\mathcal{U}$. For any $\dot{e} \in \mathcal{A}$, $\mathcal{F}(\dot{e})$ is considered the set of $\dot{e}$ – elements of the soft set $(\mathcal{F}, \mathcal{A})$. It is worthy of note that notations for soft sets could be in any other form, such as $\mathcal{F}_{\mathcal{A}}, (\mathcal{F}_{\mathcal{A}}, \mathcal{E})$ and so on.

In this work however, we shall adopt the use of these notations interchangeably.

Example 2.2. Suppose a universe $\mathcal{U}$ is the set of six cars in a gallery given by $\mathcal{U} = \{\overline{c_1}, \overline{c_2}, \overline{c_3}, \overline{c_4}, \overline{c_5}, \overline{c_6}\}$ and $\mathcal{A} = \{\dot{e}_1, \dot{e}_2, \dot{e}_3, \dot{e}_4\} \subseteq \mathcal{E}$ is the set of attributes of the cars. Let each $\dot{e}_\iota$, for $i = 1,2,3,4$ depict “high tech”, “Affordable” “Attractive” and “Safety” respectively. A buyer visiting the gallery is at liberty to construct a soft set $(\mathcal{F}_{\mathcal{A}}, \mathcal{E})$ which represent the attributes of his/her choice of vehicle.

Suppose a certain buyer, Dr. Ugorich choices are: $\mathcal{F}_{\mathcal{A}}(\dot{e_1}) = \{c_2, c_4, c_5, c_6\}$, $\mathcal{F}_{\mathcal{A}}(\dot{e_2}) = \{c_1, c_2, c_3, c_4, c_5\}$, $\mathcal{F}_{\mathcal{A}}(\dot{e_3}) = \{c_1, c_3, c_4, c_6\}$ and $\mathcal{F}_{\mathcal{A}}(\dot{e_4}) = \{c_1, c_2, c_4, c_5, c_6\}$. Then the soft set $(\mathcal{F}_A, \mathcal{E})$ is the collection of approximations: $\{\mathcal{F}_{\mathcal{A}}(e_i), i = 1,2,3,4\}$ of subset of U given by

$$\mathcal{F}_{\mathcal{A}} = \begin{cases} (\dot{e}_1, \{\overline{c_2}, \overline{c_4}, \overline{c_5}, \overline{c_6}\}),\ (\dot{e}_2, \{\overline{c_1}, \overline{c_2}, \overline{c_3}, \overline{c_4}, \overline{c_5}\}) \\ (\dot{e}_3, \{\overline{c_1}, \bar{c}_3, \bar{c}_4, \overline{c_6}\}),\ (\dot{e}_4, \{\overline{c_1}, \bar{c}_2, \overline{c_4}, \overline{c_5}, \overline{c_6}\}) \end{cases}$$

where for instance $\mathcal{F}_{\mathcal{A}}(\dot{e_3})$ means cars (attractive), whose functional value, called the $\dot{e}_3$ - approximate value set, is the set $\{\bar{c}_1, \bar{c}_3, \bar{c}_4, \bar{c}_6\}$. Hence, each approximation in the soft set $\mathcal{F}_{\mathcal{A}}$ has two parts;

i) A predicate $\mathcal{F}_{\mathcal{A}}(\dot{e}_i), i = 1,2,3,4$.

ii) The approximate set $\{\overline{c_2}, \overline{c_4}, \bar{c}_5, \overline{c_6}\}$, $\{\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4, \overline{c_5}\}$, $\{\overline{c_1}, \bar{c}_3, \overline{c_4}, \overline{c_6}\}$, $\{\overline{c_1}, \bar{c}_2, \overline{c_4}, \overline{c_5}, \overline{c_6}\}$

Definition 2.3. Let $(\mathcal{F}_{\mathcal{A}}, \mathcal{E})$ be a soft set over $\mathcal{U}$. Then $(\mathcal{F}_{\mathcal{A}}, \mathcal{E})$ is a null soft set if $\mathcal{F}_{\mathcal{A}}(\dot{e}) = \emptyset\ \forall\ \dot{e} \in \mathcal{A}$.

Definition 2.4. Suppose $(\mathcal{F}, \mathcal{A})$and $(\mathcal{G}, \mathcal{B})$ are soft sets over the universe $\mathcal{U}$. Then $(\mathcal{F}, \mathcal{A})$ is a soft subset of $(\mathcal{G}, \mathcal{B})$ if $\mathcal{A} \sqsubseteq \mathcal{B}$ and for all $\dot{e} \in, \mathcal{F}_{\mathcal{A}}(\dot{e}) = \mathcal{G}_{\mathcal{B}}(\dot{e})$.

Definition 2.5. [9] . Suppose $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ are soft sets. Then their union over the universe $\mathcal{U}$ is denoted by $(\mathcal{F}, \mathcal{A})\ \breve{\cup}\ (\mathcal{G}, \mathcal{B}) = (\mathcal{H}, \mathcal{C})$ where $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$, and $\forall\ c \in \mathcal{C}, (\mathcal{H}, \mathcal{C})$ is defined by

$$\mathcal{H}(\dot{e}) = \begin{cases} \mathcal{F}(\dot{e}) & \text{if } c \in \mathcal{A} - \mathcal{B} \\ \mathcal{G}(\dot{e}) & \text{if } c \in \mathcal{B} - \mathcal{A} \\ \mathcal{F}(\dot{e}) \cup \mathcal{G}(\dot{e}) & \text{if } c \in \mathcal{A} \cap \mathcal{B} \end{cases}$$

Definition 2.6 [2] Suppose $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ are soft sets over the universe $\mathcal{U}$. Then the restricted union, designated by $(\mathcal{F}, \mathcal{A}) \cup_R (\mathcal{G}, \mathcal{B})$ is the soft set $(\mathcal{H}, \mathcal{C})$, where $\mathcal{C} = \mathcal{A} \cap \mathcal{B}$ and $\forall\, c \in C,\ H(e) = F(e) \cup G(e)$. The restricted intersection is defined similarly.

Definition 2.7 [9]. Suppose (F, A) and (G, B) are soft sets over the universe $\mathcal{U}$. Then we have that

i. $(\mathcal{F}, \mathcal{A}) \wedge (\mathcal{G}, \mathcal{B})$ is a soft set defined by $(\mathcal{F}, \mathcal{A}) \wedge (\mathcal{G}, \mathcal{B}) = (\mathcal{H},\ \mathcal{A} \times \mathcal{B})$, where $H(\tilde{\alpha}, \tilde{\beta}) = \mathcal{F}(\tilde{\alpha}) \cap \mathcal{G}(\tilde{\beta})\ \forall\ (\tilde{\alpha}, \tilde{\beta}) \in \mathcal{A} \times \mathcal{B}$, where $\cap$ is the intersection operation of sets.

ii. $(\mathcal{F}, \mathcal{A}) \vee (\mathcal{G}, \mathcal{B})$ is a soft set defined by $(\mathcal{F}, \mathcal{A}) \vee (\mathcal{G}, \mathcal{B}) = (\mathcal{K},\ \mathcal{A} \times \mathcal{B})$, where $\mathcal{K}(\tilde{\alpha}, \tilde{\beta}) = \mathcal{F}(\tilde{\alpha}) \cup \mathcal{G}(\tilde{\beta})$ for all $(\tilde{\alpha}, \tilde{\beta}) \in \mathcal{A} \times \mathcal{B}$, where $\cup$ is the union operation of sets.

The following Proposition is a result obtained in [9]

Proposition 2.8 [9]. Suppose $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ are soft sets over the universe $\mathcal{U}$. Then

i. $\left((\mathcal{F}, \mathcal{A}) \cup (\mathcal{G}, \mathcal{B})\right)^c = (\mathcal{F}, \mathcal{A})^c \cup (\mathcal{G}, \mathcal{B})^C$

ii. $\left((\mathcal{F}, \mathcal{A}) \cap (\mathcal{G}, \mathcal{B})\right)^c = (\mathcal{F}, \mathcal{A})^c \cap (\mathcal{G}, \mathcal{B})^C$

Definition 2.9. [18]. Suppose $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ are soft sets over the universe $\mathcal{U}$. Then their extended intersection is the soft set $(\mathcal{H}, \mathcal{C})$ where $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ and $\forall c \in \mathcal{C}$, we have that

$$\mathcal{H}(c) = \begin{cases} \mathcal{F}(c) & \text{if } c \in \mathcal{A} - \mathcal{B} \\ \mathcal{G}(c) & \text{if } c \in \mathcal{B} - \mathcal{A} \\ \mathcal{F}(c) \cup \mathcal{G}(c) & \text{if } c \in \mathcal{A} \cap \mathcal{B} \end{cases}$$

We write $(\mathcal{F}, \mathcal{A}) \cap_\epsilon (\mathcal{G}, \mathcal{B}) = (\mathcal{H}, \mathcal{C})$. The extended union is defined similarly.

It is interesting to know that the De Morgan's law also holds for the extended intersection and union.

Theorem 2.10. [18]. Suppose $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ are soft sets over the universe $\mathcal{U}$. Then

i. $\left((\mathcal{F}, \mathcal{A}) \cup_\epsilon (\mathcal{G}, \mathcal{B})\right)^c = (\mathcal{F}, \mathcal{A})^c \cap_\epsilon (\mathcal{G}, \mathcal{B})^C$

ii. $\left((\mathcal{F}, \mathcal{A}) \cap_\epsilon (\mathcal{G}, \mathcal{B})\right)^c = (\mathcal{F}, \mathcal{A})^c \cup_\epsilon (\mathcal{G}, \mathcal{B})^C$

Lemma 2.11. [18]. Suppose $(\mathcal{F},\mathcal{A}),(\mathcal{G},\mathcal{B})$ and $(\mathcal{H},\mathcal{C})$ are soft sets over the universe $\mathcal{U}$. Then the following statement holds.

$$\big((\mathcal{F},\mathcal{A})\cap_R(\mathcal{G},\mathcal{B})\big)\cup_\epsilon\ (\mathcal{H},\mathcal{C})=\big((\mathcal{F},\mathcal{A})\cup_\epsilon(\mathcal{H},\mathcal{C})\big)\cap_R\big((\mathcal{G},\mathcal{B})\cup_\epsilon(\mathcal{H},\mathcal{C})\big)$$

where $\mathcal{A}\cap\mathcal{B}\neq\emptyset$.

3. Main Results

We characterize soft sets as soft semigroups in this segment and present some semigroup properties associated with the hybrid algebraic structure.

Let S be a semigroup and let $\mathcal{E}=\{\dot{e_1},\dot{e_2},\dots,\dot{e_n}\}$ be a set of attributes (parameters). Let $A\subseteq E$ and let $f:\mathcal{A}\to\mathcal{P}(S)$ be a mapping, where $\mathcal{P}(S)$ is the set of all subsets of S. Then a non empty soft set $(\mathcal{F},\mathcal{A})$ over S is called a soft semigroup if and only if $f(a)$ is a subsemigroup of S, for each $a\,\epsilon\,\mathcal{A}$. i.e $h,y\,\epsilon\,f(a)\Longrightarrow xy\,\epsilon\,f(a)$.

Let S be a semigroup and let $\mathcal{E}$ be a set of attributes. Let $\mathcal{A}\subseteq E$ and Let $f:\mathcal{A}\to\mathcal{P}(S)$ be a mapping, where $\mathcal{P}(S)$ is the set of all subsets of S. Then $(\mathcal{F},\mathcal{A})$ is a soft ideal over S if and only if $f(a)$ is an ideal of S for each $a\,\epsilon\,A$. i.e $h\,\epsilon\,f(a),\ r\,\epsilon\,S\Longrightarrow rh\,\epsilon\,f(a)$ and $hr\,\epsilon\,f(a)$.

Example 3.1. Consider $S=\{\dot{a},\dot{b},\dot{c},\dot{d}\}$ be a semigroup defined by the Cayley's table below

$\star$	$\dot{a}$	$\dot{b}$	$\dot{c}$	$\dot{d}$
$\dot{a}$	$\dot{a}$	$\dot{a}$	$\dot{a}$	$\dot{a}$
$\dot{b}$	$\dot{a}$	$\dot{a}$	a	$\dot{a}$
$\dot{c}$	$\dot{a}$	$\dot{a}$	$\dot{b}$	$\dot{a}$
$\dot{d}$	$\dot{a}$	$\dot{a}$	$\dot{b}$	$\dot{b}$

Define $\mathcal{F}:S\to\mathcal{P}(S)$ by $\mathcal{F}(\dot{a})=\{\dot{a}\}$, $\mathcal{F}(\dot{b})=\{\dot{a},\dot{b}\}$, $\mathcal{F}(\dot{c})=\{\dot{a},\dot{b},\dot{c}\}$, $\mathcal{F}(\dot{d})=\{\dot{a},\dot{b},\dot{d}\}$. Then $(\mathcal{F},S)$ is a soft set and a soft semigroup over S, since $\mathcal{F}(\dot{x})$ is a subsemigroup of S for $\forall\,\dot{x}\,\epsilon\,S$.

It is pertinent we note that it is not always true that every soft set over a semigroup S, is a soft semigroup over S.

We illustrate this fact as follows;

Suppose $(\mathcal{G}, S)$ is a soft set such that $\mathcal{G}: S \to \mathcal{P}(S)$ is defined as $\mathcal{G}(\dot{b}) = \{\dot{b}\}$. Then it is clear that $(\mathcal{G}, S)$ is not a soft semigroup over *S* since $\mathcal{G}(\dot{b}) = \{\dot{b}\}$ is not a subsemigroup of *S*.

The following results give some properties of soft semigroups

Lemma 3.2. Let $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ be soft semigroups over a semigroup *S*. Then the restricted intersection $(\mathcal{F}, \mathcal{A}) \cap_R (\mathcal{G}, \mathcal{B})$ is also a soft semigroup provided that it is non empty.

Proof. We know that $(\mathcal{F}, \mathcal{A}) \cap_R (\mathcal{G}, \mathcal{B}) = (\mathcal{H}, \mathcal{C})$ where $\mathcal{C} = \mathcal{A} \cap \mathcal{B} \neq \emptyset$ and $\mathcal{H}(c) = \mathcal{F}(c) \cap \mathcal{G}(c)\ \forall\ c \in C$. Obviously, it is either empty or a subsemigroup of *S*. Thus, $(\mathcal{H}, \mathcal{C})$ is a soft semigroup over *S*.

Lemma 3.3. Suppose $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ are two soft semigroups over a semigroup *S* with $A \cap B \neq \emptyset$. Then $(\mathcal{F}, \mathcal{A}) \cup_\epsilon (\mathcal{G}, \mathcal{B})$, their extended union is also a soft semigroup over *S*.

Proof. We know that $(\mathcal{F}, \mathcal{A}) \cup_\epsilon (\mathcal{G}, \mathcal{B}) = (\mathcal{H}, \mathcal{C})$ and $\mathcal{A} \cap \mathcal{B} \neq \emptyset$, for all $c \in \mathcal{C} = \mathcal{A} \cup \mathcal{B}$ either $c \in \mathcal{A} - \mathcal{B}$ or $c \in \mathcal{B} - \mathcal{A}$. Now if $c \in \mathcal{A} - \mathcal{B}$ then $\mathcal{H}(c) = \mathcal{F}(c)$, and if $c \in \mathcal{B} - \mathcal{A}$ then $\mathcal{H}(c) = \mathcal{G}(c)$. But in both cases, $\mathcal{H}(c)$ is a subsemigroup of *S*. Thus, $(\mathcal{H}, \mathcal{C})$ is a soft semigroup of *S*.

Lemma 3.4. Suppose $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ are soft ideals over *S*. Their restricted intersection $(\mathcal{F}, \mathcal{A}) \cap_R (\mathcal{G}, \mathcal{B})$ is also a soft ideal over *S*, which is contained in $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ whenever $(\mathcal{F}, \mathcal{A}) \cap_R (\mathcal{G}, \mathcal{B}) = \emptyset$.

Proof. Obviously, $(\mathcal{F}, \mathcal{A}) \cap_R (\mathcal{G}, \mathcal{B}) = (\mathcal{H}, \mathcal{C})$ where $\mathcal{C} = \mathcal{A} \cap \mathcal{B} \neq \emptyset$ and $\mathcal{H}(c) = \mathcal{F}(c) \cap \mathcal{G}(c)$ is either empty or an ideal of *S*. Thus $(\mathcal{H}, \mathcal{C})$ is a soft ideal over *S*.
It can be easily seen that $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{B}$. Moreso, $\mathcal{H}(c) \subseteq \mathcal{F}(c)$ and $\mathcal{H}(c) \subseteq \mathcal{G}(c)$ so that $(\mathcal{H}, \mathcal{C}) \subseteq (\mathcal{F}, \mathcal{A})$ and $(\mathcal{H}, \mathcal{C}) \subseteq (\mathcal{G}, \mathcal{B})$.

The lemma above is also applicable for the extended union as shown below

Lemma 3.5. Let $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ are soft ideals over *S*. Their extended union, $(\mathcal{F}, \mathcal{A}) \cup_\epsilon (\mathcal{G}, \mathcal{B})$ is also a soft ideal over *S* containing $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$.

Proof. Since $(\mathcal{F}, \mathcal{A}) \cup_\epsilon (\mathcal{G}, \mathcal{B}) = (\mathcal{H}, \mathcal{C})$, we have that $\forall\ c \in \mathcal{C} = \mathcal{A} \cup \mathcal{B}$ either $c \in \mathcal{A} - \mathcal{B}$ or $c \in \mathcal{B} - \mathcal{A}$. If $c \in \mathcal{A} - \mathcal{B}$, then $\mathcal{H}(c) = \mathcal{F}(c)$ and if $c \in \mathcal{B} - \mathcal{A}$ then $\mathcal{H}(c) = \mathcal{G}(c)$. Now, if $c \in \mathcal{A} \cap \mathcal{B}$, then $\mathcal{H}(c) = \mathcal{F}(c) \cup \mathcal{G}(c)$. In any case, $\mathcal{H}(c)$ is an ideal of *S*.
Consequently, $(\mathcal{H}, \mathcal{C})$ is a soft ideal over *S*.

Obviously, $(\mathcal{F},\mathcal{A}) \subseteq (\mathcal{H},\mathcal{C})$ and $(\mathcal{G},\mathcal{B}) \subseteq (\mathcal{H},\mathcal{C})$.

The next result shows that if we let the parameter $\mathcal{A}$ to be fixed, then the distributive law holds for soft ideals over *S*.

Theorem 3.6. Let $(\mathcal{F},\mathcal{A})$, $(\mathcal{G},\mathcal{B})$ and $(\mathcal{H},\mathcal{A})$ be soft ideals over *S*. Then the following statement holds.

$$\big((\mathcal{F},\mathcal{A}) \cup_R (\mathcal{G},\mathcal{A})\big) \cap_R (\mathcal{H},\mathcal{A}) = \big((\mathcal{F},\mathcal{A}) \cap_R (\mathcal{H},\mathcal{A})\big) \cup_R \big((\mathcal{G},\mathcal{A}) \cap_R (\mathcal{H},\mathcal{A})\big)$$

Proof. From the LHS, we have that

$$\big((\mathcal{F},\mathcal{A}) \cup_R (\mathcal{G},\mathcal{A})\big) \cap_R (\mathcal{H},\mathcal{A}) = (\mathcal{M},\mathcal{A}) \cap_R (\mathcal{H},\mathcal{A})$$

where $(\mathcal{F},\mathcal{A}) \cup_R (\mathcal{G},\mathcal{A}) = (\mathcal{M},\mathcal{A})$ and $\mathcal{M}(\dot{a}) = \mathcal{F}(\dot{a}) \cup \mathcal{G}(\dot{a})$.

So that $(\mathcal{M},\mathcal{A}) \cap_R (\mathcal{H},\mathcal{A}) = (\mathcal{N},\mathcal{A})$ and

$$\begin{aligned}\mathcal{N}(\dot{a}) &= \mathcal{M}(\dot{a}) \cap \mathcal{H}(\dot{a}) \\ &= \big(\mathcal{F}(\dot{a}) \cup \mathcal{G}(\dot{a})\big) \cap \mathcal{H}(\dot{a}) \\ &= \big(\mathcal{F}(\dot{a}) \cap \mathcal{H}(\dot{a})\big) \cup \big(\mathcal{G}(\dot{a}) \cap \mathcal{H}(\dot{a})\big).\end{aligned}$$

From the RHS, we have that

$$(\mathcal{F},\mathcal{A}) \cap_R (\mathcal{H},\mathcal{A}) = (\mathcal{P},\mathcal{A}) \text{ and } P(\dot{a}) = F(\dot{a}) \cap H(\dot{a}).$$

Similarly, $(\mathcal{G},\mathcal{A}) \cap_R (\mathcal{H},\mathcal{A}) = (\mathcal{Q},\mathcal{A})$ and $\mathcal{Q}(\dot{a}) = \mathcal{G}(\dot{a}) \cap \mathcal{H}(\dot{a})$.

Consequently, we have that

$$\mathcal{M}(\dot{a}) \cap \mathcal{H}(\dot{a}) = \mathcal{N}(\dot{a}) = \mathcal{P}(\dot{a}) \cup \mathcal{Q}(\dot{a}).$$

Thus, $\big((\mathcal{F},\mathcal{A}) \cup_R (\mathcal{G},\mathcal{A})\big) \cap_R (\mathcal{H},\mathcal{A}) = \big((\mathcal{F},\mathcal{A}) \cap_R (\mathcal{H},\mathcal{A})\big) \cup_R \big((\mathcal{G},\mathcal{A}) \cap_R (\mathcal{H},\mathcal{A})\big)$.

Having considered soft ideals, we now present the concept of homomorphisms between soft semigroups namely; soft homomorphisms.

Let S and T be two semigroups and let $(\mathcal{F},\mathcal{A})$ and $(\mathcal{G},\mathcal{B})$ are soft semigroups such that $\alpha : S \to T$ and $\beta : \mathcal{A} \to \mathcal{B}$ are two functions. Then (α,β) is a soft homomorphism if the following conditions are satisfied;

i. α is an epimorphism from S onto T
ii. β is surjective from $\mathcal{A}$ to $\mathcal{B}$
iii. $\alpha(\mathcal{F}(\dot{a})) = \mathcal{G}(\beta(\dot{a}))\ \forall\ a \epsilon A$.

In this case, $(\mathcal{F},\mathcal{A})$ is said to be soft homomorphic to $(\mathcal{G},\mathcal{B})$.

It is important to note that if $\alpha : S \to T$ is an isomorphism and $\beta : \mathcal{A} \to \mathcal{B}$ is bijective, then (α,β) is called a soft isomorphism and we denote this by $(F,A) \cong (G,B)$.

Lemma 3.7. Let S and T be two semigroups and let $(\mathcal{F},\mathcal{A})$ and $(\mathcal{G},\mathcal{B})$ be soft semigroups such that $(\alpha,\beta):(\mathcal{F},\mathcal{A})\to(\mathcal{G},\mathcal{B})$ is a soft homomorphism. If $(\mathcal{F},\mathcal{A})$ is soft ideal over S, then $(\mathcal{G},\mathcal{B})$ is a soft ideal over T.

Proof. Obviously, $(\mathcal{F},\mathcal{A})$ is a soft ideal over S, which implies that $\mathcal{F}(\dot{a})$ is an ideal of S. Now since (α,β) is a soft homomorphism, then we have that for each $b\,\epsilon\,\mathcal{B}$ there exists $\dot{a}\,\epsilon\,A$ such that $\beta(\dot{a})=\dot{b}$.

Consequently, we have that

$$\mathcal{G}(\dot{b})=\mathcal{G}(\beta(\dot{a}))=\alpha(\mathcal{F}(\dot{a}))\,.$$

Since $\mathcal{F}(\dot{a})$ is an ideal of S, this implies that $\alpha(\mathcal{F}(\dot{a}))$ is an ideal of $\mathcal{H}\big(\beta(\dot{b})\big)$.

Thus, $\mathcal{G}(\dot{b})$ is an ideal of $\mathcal{H}\big(\beta(\dot{b})\big)\ \forall\ \dot{b}\,\epsilon\,\mathcal{B}$.

Hence $(\mathcal{G},\mathcal{B})$ is a soft ideal over T.

Lemma 3.8. Let S and T be two semigroups and let $(\mathcal{F},\mathcal{A})$ and $(\mathcal{G},\mathcal{B})$ soft semigroups, such that $(\mathcal{G},\mathcal{B})$ is a soft ideal of $(\mathcal{F},\mathcal{A})$ over S. Then for a soft semigroup $(\mathcal{H},\mathcal{C})$ over T, $(\alpha(\mathcal{G}),\ \beta(\mathcal{B}))$ is a soft ideal of $(\mathcal{H},\mathcal{C})$ and $(\alpha,\beta):(\mathcal{F},\mathcal{A})\to(\mathcal{H},\mathcal{C})$ is a soft homomorphism.

Proof. It can be easily seen that $\mathcal{G}(\dot{b})$ is an ideal of $\mathcal{F}(\dot{b})\ \forall\ \dot{b}\,\epsilon\,\mathcal{B}$, so we have that $\mathcal{G}(\dot{b})\subseteq\mathcal{F}(\dot{b})$ which implies that $\alpha\left(\mathcal{G}(\dot{b})\right)\subseteq\alpha\left(\mathcal{F}(\dot{b})\right)=\mathcal{H}\left(\beta(\dot{b})\right)$.

Obviously, $\alpha\left(\mathcal{G}(\dot{b})\right)$ is an ideal of $\mathcal{T}$ since $\mathcal{G}(\dot{b})$ is an ideal of $\mathcal{F}(\dot{b})$.

Consequently, $\beta(\mathcal{B})\subseteq\mathcal{C}$ since β is a function from $\mathcal{A}$ onto $\mathcal{C}$.

Thus, $(\alpha(\mathcal{G}),\ \beta(\mathcal{B}))$ is a soft ideal of $(\mathcal{H},\mathcal{C})$.

Remark 3.9. Suppose $(\alpha,\beta):(\mathcal{F},\mathcal{A})\to(\mathcal{G},\mathcal{B})$ and $(\gamma,\theta):(\mathcal{G},\mathcal{B})\to(\mathcal{H},\mathcal{C})$ are soft homomorphisms, then the soft composition of (α,β) and (γ,θ) is defined as

$$(\alpha,\beta)\circ(\gamma,\theta)=(\Gamma,\lambda)\text{ where }\Gamma=\alpha\circ\gamma\text{ and }\lambda=\beta\circ\theta.$$

We conclude this section by characterizing soft regular semigroups. It is known that an element a of a semigroup S is said to be regular if there exist an element $x\,\epsilon\,S$ such that $\dot{a}x\dot{a}=\dot{a}$. If every element of a semigroup S is regular then S is said to be a regular semigroup.

Now with our knowledge of soft ideals and regular semigroups, we present the following result.

Theorem 3.10. Let $(\mathcal{F},\mathcal{A})$ and $(\mathcal{G},\mathcal{B})$ be two soft semigroups over a semigroup S and define the operation $\circledast$ as $(\mathcal{F},\mathcal{A}) \circledast (\mathcal{G},\mathcal{B}) = (\mathcal{H},\mathcal{A}\times\mathcal{B})$, where $\mathcal{H}(\dot{a},\dot{b}) = \mathcal{F}(\dot{a}) \circledast \mathcal{G}(\dot{b})$, $\dot{a}\in\mathcal{A}$, $\dot{b}\in\mathcal{B}$ and $\mathcal{A}\times\mathcal{B}$ is the Cartesian product of $\mathcal{A}$ and $\mathcal{B}$. Then S is a regular semigroup if and only if $(R,\mathcal{A}) \circledast (L,\mathcal{B}) = (R,\mathcal{A}) \wedge (L,\mathcal{B})$ for every soft left ideal $(L,\mathcal{B})$ and soft right ideal $(R,\mathcal{A})$ over S.

Proof. For the direct part of the proof, we know that $(R,A) \circledast (L,B) = (\mathcal{H},A\times B)$ where $\mathcal{H}$ is a function $\mathcal{A}\times\mathcal{B}$ to $P(S)$ defined by $\mathcal{H}(\dot{a},\dot{b}) = R(\dot{a}) \circledast L(\dot{b})$.
Obviously, $(R,\mathcal{A}) \wedge (L,\mathcal{B}) = (\mathcal{K},\mathcal{A}\times\mathcal{B})$, where $\mathcal{K}$ is a function from $\mathcal{A}\times\mathcal{B}$ to $\mathcal{P}(S)$ defined by $\mathcal{K}(\dot{a},\dot{b}) = R(\dot{a}) \cap L(\dot{b})$.
Consequently, $\mathcal{A}\times\mathcal{B} \subseteq \mathcal{A}\times\mathcal{B}$ and we have that

$$R(\dot{a}) \circledast L(\dot{b}) \subseteq R(\dot{a}) \circledast S \subseteq R(\dot{a}) \text{ and } R(\dot{a}) \circledast L(\dot{b}) \subseteq S \circledast L(\dot{b}) \subseteq L(\dot{b}).$$

Thus, $R(\dot{a}) \circledast L(\dot{b}) \subseteq R(\dot{a}) \cap L(\dot{b})$ for all $\dot{a}\in\mathcal{A}, \dot{b}\in\mathcal{B}$ so that $(\mathcal{H},\mathcal{A}\times\mathcal{B}) \subseteq (\mathcal{K},\mathcal{A}\times\mathcal{B})$.
Now let $x\in R(\dot{a}) \cap L(\dot{b})$. Since S is regular and $x\in S$, then there exists $y\in S$ such that $xyx = x$. Since $x\in R(\dot{a})$ and $yx\in L(\dot{b}), xyx = x \in R(\dot{a}) \circledast L(\dot{b})$ which implies that $(R,\mathcal{A}) \wedge (L,\mathcal{B}) \subseteq (R,\mathcal{A}) \circledast (L,\mathcal{B})$. Hence, $(\mathcal{K},\mathcal{A}\times\mathcal{B}) \subseteq (\mathcal{H},\mathcal{A}\times\mathcal{B})$ so that $(R,\mathcal{A}) \circledast (L,\mathcal{B}) = (R,\mathcal{A}) \wedge (L,\mathcal{B})$.
Conversely, let $\mathcal{A} = \mathcal{B} = S$ and R be a function from $\mathcal{A}$ to $\mathcal{P}(S)$. Define $R(x) = xS^1$, for all $x\in S$ and let L be a function from $\mathcal{B}$ to $\mathcal{P}(S)$, defined by $L(x) = S^1x$, for all $x\in S$. This implies that (R,S) is a soft right ideal and (L,S) is a soft left ideal over S.
So we have that $x\in R(x) \cap L(x) = R(x)L(x) = xS^1S^1x \subseteq xS^1x$.
Hence, we have that S is a regular semigroup and the theorem is proved.

Let S be a semigroup and $(\mathcal{F},\mathcal{A})$ a soft semigroup over S. Then $(\mathcal{F},\mathcal{A})$ is called a soft regular semigroup if for each $\dot{a}\in\mathcal{A}$, $\mathcal{F}(\dot{a})$ is a regular subsemigroup of S.
It is worthy of note that regularity of a soft semigroup does not necessarily imply regularity of the semigroup. This is shown in the example below.

Example 3.11. Consider $S = \{\dot{m},\dot{n},\dot{o},\dot{p}\}, S$ is a semigroup defined by the Cayley's table below

$\dot{\star}$	$\dot{m}$	$\dot{n}$	$\dot{o}$	$\dot{p}$
$\dot{m}$	$\dot{m}$	$\dot{m}$	$\dot{m}$	$\dot{m}$
$\dot{n}$	$\dot{m}$	$\dot{m}$	$\dot{m}$	$\dot{m}$
$\dot{o}$	$\dot{m}$	$\dot{m}$	$\dot{m}$	$\dot{m}$

$\dot{p}$	$\dot{m}$	$\dot{m}$	m	$\dot{p}$

It could be easily seen that S is not regular. Now, suppose $\mathcal{A} = \{x, y\}$ is a set of parameters such that $\mathcal{F}(x) = \{\dot{m}\}$, $\mathcal{F}(y) = \{\dot{m}, \dot{p}\}$. Then $(\mathcal{F}, \mathcal{A})$ is a soft regular semigroup over S since both $\mathcal{F}(x)$ and $\mathcal{F}(y)$ are regular subsemigroups of S.

4. Conclusion

In this work, we have presented some semigroup properties of soft sets other than the ones in the literature. We have also revisited some fundamental operations in the theory of soft sets and proved some new results. As a contribution to the advancement of semigroup theory, defining some new concept can be considered as positive.

This paper also motivates future research especially as regards to applications of soft semigroups.

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Details of the AI usage are given below:

1.

2.

3.

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