Derivation of Continuous Linear Multistep Hybrid Block Method for the Integration of Volterra Integral Equation of Second Kind

Abstracts

In this paper, we proposed a continuous linear multistep hybrid block method for the integration of Volterra integral equation of second kind of the form$y(x) = f(x) + \int_{x_0}^{x_n} \varphi\big(x,s)y(s)\big)ds$, using power series and trigonometrically fitted function as the trial solution for the approximation via collocation techniques. The proposed hybrid block scheme is found to be consistent, zero-stable and convergent. The implementation of the scheme on numerical problem and comparison of result obtained with existing numerical method will be included

Keywords: Multistep hybrid block method, power series, collocation and interpolation method, second kind of Volverra integral equations.

1. Introduction

Volterra integral equation is a special kind of integral equation which is classified into three: the first, second and third kind. In this research, wedevelop two off - grid points of hybrid block method for the solution ofsecond kind of Volterra integral equation due to its characteristics and uniqueness. In the literature, the second kind of Volterra integral equation (VIE) according to [1] is a form: $y(x) = f(x) + \int_{x_0}^{x_n} \varphi\big(x,s)y(s)\big)ds$ (1)

Where $f(x)$ is a given function and $\varphi(x,s)$is called the kernel of integral equation.Volterra integral equation (VIE) appears especially when we are trying to transform an initial value problem into integral form, so that, the solution of the equation can be easily obtained than the original initial value problem [2].Solving (1) is equivalent to solving the following initial value problem for ordinary differential equations of the first order

$$y'(x) = f'(x) + \varphi\big(x,y(x)\big),\ y(x_0) = f(x_0) \quad (2)$$

The Volterra Integral Equations are widely used in population growth models, physics, chemistry, and engineering [3]. Particularly important in science and engineering are systems of linear integral equations and their precise or approximate solutions. In the domains of engineering and applied research, some of these integral equations cannot be solved explicitly, hence approximation or numerical methods must frequently be used [4]. In recent years, many strategies for resolving Volterra integral equations are suggested by many researchers such as: [5], [6], [7], [8], and much recently by [9], [10], [11] and [12].

2.**Derivation of the proposed method**

In this section, we derive three step with two off-grid points ofhybrid block method for the integration of Volterra integral equation of second kind by carefully selecting $p=\frac{1}{3}$ and $q=\frac{2}{3}$ for $p, q \in [0,1]$

Let the approximate series solution and trigonometrically fitted function of the Eq. (1) takes the form of $z(x)=\sum_{j=0}^{3}\varphi_j x^j + \sum_{j=1}^{2}\lambda_j \sin x + \sum_{j=1}^{2}\lambda_j \cos x$ (3)

Where $\emptyset_j$and λ_j aer the coefficients to be determined.

Consider the ordinary differential equation

$$z''= f(x,z,z'), z(a_0)=z_0 \; , \; z'(a)=z'_0 \qquad (4)$$

Subject to the condition

$$z(x)= y(x)- f(x) \qquad (5)$$

The second derivative of Eq. (3) is given as;

$$z''(x)=\sum_{j=0}^{3} j(j-1)\phi_j x^{j-2} - \sum_{j=1}^{2}\lambda_j(\sin x) - \sum_{j=1}^{2}\lambda_j(\cos x) \qquad (6)$$

Substituting Eq.(4) into (1) gives

$$g(x,z,z')=\sum_{j=0}^{3} j(j-1)\varphi_j x^{j-2} - \sum_{j-1}^{2}\lambda_j \sin x - \sum_{j=1}^{2}\lambda_j \cos x \qquad (7)$$

Interpolating (3) at $x_{n+\varpi}, \varpi = 0,1$ and collocating (6) at $x_{n+\vartheta,} \vartheta = \{0,\frac{1}{3},\frac{2}{3},1,2,3\}$ leads to the system of nonlinear equations written in the form

$z_n(x) = X(x)A$ (8)

$$\begin{bmatrix} 3 & 4x_n & -\frac{3}{2}x_n^2 & -\frac{1}{2}x_n^3 & \frac{17}{24}x_n^4 & \frac{11}{40}x_n^5 & -\frac{4}{45}x_n^6 & -\frac{8}{315}x_n^7 \\ 3 & (4x_n+4h) & -\frac{3}{2}(x_n+h)^2 & -\frac{1}{2}(x_n+h)^3 & \frac{17}{24}(x_n+h)^4 & \frac{11}{40}(x_n+h)^5 & -\frac{4}{45}(x_n+h)^6 & -\frac{8}{315}(x_n+h)^7 \\ 0 & 0 & -3 & -3x_n & \frac{17}{2}x_n^2 & \frac{11}{2}x_n^3 & -\frac{8}{3}x_n^4 & -\frac{16}{15}x_n^5 \\ 0 & 0 & -3 & -3x_n-h & \frac{17}{2}\left(x_n+\frac{1}{3}h\right)^2 & \frac{11}{2}\left(x_n+\frac{1}{3}h\right)^3 & -\frac{8}{3}\left(x_n+\frac{1}{3}h\right)^4 & -\frac{16}{15}\left(x_n+\frac{1}{3}h\right)^5 \\ 0 & 0 & -3 & -3x_n-2h & \frac{17}{2}\left(x_n+\frac{2}{3}h\right)^2 & \frac{11}{2}\left(x_n+\frac{2}{3}h\right)^3 & -\frac{8}{3}\left(x_n+\frac{2}{3}h\right)^4 & -\frac{16}{15}\left(x_n+\frac{2}{3}h\right)^5 \\ 0 & 0 & -3 & -3x_n-3h & \frac{17}{2}(x_n+h)^2 & \frac{11}{2}(x_n+h)^3 & -\frac{8}{3}(x_n+h)^4 & -\frac{16}{15}(x_n+h)^5 \\ 0 & 0 & -3 & -3x_n-6h & \frac{17}{2}(x_n+2h)^2 & \frac{11}{2}(x_n+2h)^3 & -\frac{8}{3}(x_n+2h)^4 & -\frac{16}{15}(x_n+2h)^5 \\ 0 & 0 & -3 & -3x_n-9h & \frac{17}{2}(x_n+3h)^2 & \frac{11}{2}(x_n+3h)^3 & -\frac{8}{3}(x_n+3h)^4 & -\frac{16}{15}(x_n+3h)^5 \end{bmatrix} \begin{bmatrix} \psi_0 \\ \psi_1 \\ \phi_0 \\ \phi_{\frac{1}{3}} \\ \phi_{\frac{2}{3}} \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} z_n \\ z_{n+1} \\ g_n \\ g_{n+\frac{1}{3}} \\ g_{n+\frac{2}{3}} \\ g_{n+1} \\ g_{n+2} \\ g_{n+3} \end{bmatrix}$$

Using the Gaussian elimination method to solve Eq. (8) gives the coefficients $\psi_0, \psi_1, \phi_0, \phi_{\frac{1}{3}}, \phi_{\frac{2}{3}}, \phi_1, \phi_2, \phi_3$, which are then substituted into (3) and simplified to give the implicit second derivative hybrid block method of the form;

$$z(x) = \sum_{i=0,1} \psi_i z_{n+i} + h^2 \left[\sum_{i=\frac{1}{3},\frac{2}{3}} \phi_i g_{n+i} + \sum_{i=0}^{3} \phi_i g_{n+i} \right] \quad (9)$$

Differentiating equation (8) to give:

$$z'(x) = \frac{1}{h} \sum_{i=0,1} \psi_i z_{n+i} + h \left[\sum_{i=\frac{1}{3},\frac{2}{3}} \phi_i g_{n+i} + \sum_{i=0}^{3} \phi_i g_{n+i} \right] \quad (10)$$

Where

$\psi_0 = 1 - \dfrac{-x_n + x}{h}$:

$\psi_1 = \dfrac{-x_n + x}{h}$:

$$-\frac{27}{40}\frac{\left(-x_n+x\right)^6}{h^4}+\frac{81}{1120}\frac{\left(-x_n+x\right)^7}{h^5}$$

$$\phi_0=-\frac{32}{315}\left(-x_n+x\right)h+\frac{1}{2}\left(-x_n+x\right)^2-\frac{19}{18}\frac{\left(-x_n+x\right)^3}{h}+\frac{55}{48}\frac{\left(-x_n+x\right)^4}{h^2}-\frac{31}{48}\frac{\left(-x_n+x\right)^5}{h^3}+\frac{7}{40}\frac{\left(-x_n+x\right)^6}{h^4}-\frac{1}{56}\frac{\left(-x_n+x\right)^7}{h^5}$$

$$\phi_{\frac{1}{3}}=-\frac{729}{2240}\left(-x_n+x\right)h+\frac{81}{40}\frac{\left(-x_n+x\right)^3}{h}-\frac{27}{8}\frac{\left(-x_n+x\right)^4}{h^2}+\frac{729}{320}\frac{\left(-x_n+x\right)^5}{h^3}$$

$$\phi_{\frac{2}{3}}=-\frac{81}{1960}\left(-x_n+x\right)h-\frac{81}{56}\frac{\left(-x_n+x\right)^3}{h}+\frac{783}{224}\frac{\left(-x_n+x\right)^4}{h^2}-\frac{3159}{1120}\frac{\left(-x_n+x\right)^5}{h^3}+\frac{513}{560}\frac{\left(-x_n+x\right)^6}{h^4}-\frac{81}{784}\frac{\left(-x_n+x\right)^7}{h^5}$$

$$\phi_1=-\frac{11}{336}\left(-x_n+x\right)h+\frac{1}{2}\frac{\left(-x_n+x\right)^3}{h}-\frac{4}{3}\frac{\left(-x_n+x\right)^4}{h^2}+\frac{101}{80}\frac{\left(-x_n+x\right)^5}{h^3}-\frac{9}{20}\frac{\left(-x_n+x\right)^6}{h^4}+\frac{3}{56}\frac{\left(-x_n+x\right)^7}{h^5}$$

$$\phi_2=\frac{1}{840}\left(-x_n+x\right)h-\frac{1}{40}\frac{\left(-x_n+x\right)^3}{h}+\frac{7}{96}\frac{\left(-x_n+x\right)^4}{h^2}-\frac{13}{160}\frac{\left(-x_n+x\right)^5}{h^3}+\frac{3}{80}\frac{\left(-x_n+x\right)^6}{h^4}-\frac{3}{560}\frac{\left(-x_n+x\right)^7}{h^5}$$

$$\phi_3=-\frac{13}{141120}\left(-x_n+x\right)h+\frac{1}{504}\frac{\left(-x_n+x\right)^3}{h}-\frac{1}{168}\frac{\left(-x_n+x\right)^4}{h^2}+\frac{47}{6720}\frac{\left(-x_n+x\right)^5}{h^3}-\frac{1}{280}\frac{\left(-x_n+x\right)^6}{h^4}+\frac{1}{1568}\frac{\left(-x_n+x\right)^7}{h^5}$$

Evaluating Eq.(9) at non-interpolating points $x=x_{n+\frac{1}{3}},x_{n+\frac{2}{3}},x_{n+2},x_{n+3}$ yields

$$z_{n+\frac{1}{3}}=\frac{2}{3}z_n+\frac{1}{3}z_{n+1}-\frac{497}{87480}h^2g_n-\frac{1}{15}h^2g_{n+\frac{1}{3}}-\frac{523}{15120}h^2g_{n+\frac{2}{3}}-\frac{31}{7290}h^2g_{n+1}+\frac{1}{11664}h^2g_{n+2}-\frac{1}{153090}h^2g_{n+3}$$

$$z_{n+\frac{2}{3}}=\frac{1}{3}z_n+\frac{2}{3}z_{n+1}-\frac{29}{10935}h^2g_n-\frac{167}{4320}h^2g_{n+\frac{1}{3}}-\frac{79}{1260}h^2g_{n+\frac{2}{3}}-\frac{209}{29160}h^2g_{n+1}+\frac{1}{14580}h^2g_{n+2}-\frac{11}{2449440}h^2g_{n+3} \qquad (11)$$

$$z_{n+2} = -z_n + 2z_{n+1} - \frac{1}{15}h^2 g_n + \frac{81}{160}h^2 g_{n+\frac{1}{3}} - \frac{81}{140}h^2 g_{n+\frac{2}{3}} + \frac{127}{120}h^2 g_{n+1} + \frac{1}{12}h^2 g_{n+2} - \frac{3}{1120}h^2 g_{n+3}$$

$$z_{n+3} = -2z_n + 3z_{n+1} + \frac{11}{120}h^2 g_n + \frac{243}{560}h^2 g_{n+\frac{2}{3}} + \frac{13}{10}h^2 g_{n+1} + \frac{89}{80}h^2 g_{n+2} + \frac{13}{210}h^2 g_{n+3}$$

Evaluating Eq.(9) at all points, to obtain

$$hz'_n = -z_n + z_{n+1} - \frac{32}{315}h^2 g_n - \frac{729}{2240}h^2 g_{n+\frac{1}{3}} - \frac{81}{1960}h^2 g_{n+\frac{2}{3}} - \frac{11}{336}h^2 g_{n+1} + \frac{1}{840}h^2 g_{n+2} - \frac{13}{141120}h^2 g_{n+3}$$

$$hz'_{n+\frac{1}{3}} = -z_n + z_{n+1} + \frac{5687}{408240}h^2 g_n - \frac{13}{504}h^2 g_{n+\frac{1}{3}} - \frac{2231}{14112}h^2 g_{n+\frac{2}{3}} + \frac{127}{34020}h^2 g_{n+1} + \frac{53}{1428840}h^2 g_{n+2} - \frac{131}{272160}h^2 g_{n+3}$$

$$hz'_{n+\frac{2}{3}} = -z_n + z_{n+1} + \frac{131}{25515}h^2 g_n + \frac{547}{4032}h^2 g_{n+\frac{1}{3}} + \frac{839}{17640}h^2 g_{n+\frac{2}{3}} - \frac{2999}{136080}h^2 g_{n+1} + \frac{5}{13608}h^2 g_{n+2} - \frac{269}{11430720}h^2 g_{n+3} \qquad (12)$$

$$hz'_{n+1} = -z_n + z_{n+1} + \frac{11}{1008}h^2 g_n + \frac{27}{280}h^2 g_{n+\frac{1}{3}} + \frac{2133}{7840}h^2 g_{n+\frac{2}{3}} + \frac{17}{140}h^2 g_{n+1} - \frac{1}{1120}h^2 g_{n+2} + \frac{1}{17640}h^2 g_{n+3}$$

$$hz'_{n+2} = -z_n + z_{n+1} - \frac{53}{315}h^2 g_n + \frac{459}{448}h^2 g_{n+\frac{1}{3}} - \frac{621}{392}h^2 g_{n+\frac{2}{3}} + \frac{3193}{1680}h^2 g_{n+1} + \frac{281}{840}h^2 g_{n+2} - \frac{1021}{141120}h^2 g_{n+3}$$

$$hz'_{n+3} = -z_n + z_{n+1} + \frac{3079}{5040}h^2 g_n - \frac{729}{280}h^2 g_{n+\frac{1}{3}} + \frac{35397}{7840}h^2 g_{n+\frac{2}{3}} - \frac{817}{420}h^2 g_{n+1} + \frac{1097}{672}h^2 g_{n+2} + \frac{1025}{3528}h^2 g_{n+3}$$

This gives the following equation in matrix form

$$A_1 Z = A_2 R_1 + B_1 R_2 + B_2 R_3 \tag{13}$$

$$A_1 = \begin{bmatrix} 1 & 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{h} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{h} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{h} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{h} & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, z = \begin{bmatrix} z_{n+\frac{1}{3}} \\ z_{n+\frac{2}{3}} \\ z_{n+1} \\ z_{n+2} \\ z_{n+3} \\ z'_{n+\frac{1}{3}} \\ z'_{n+\frac{2}{3}} \\ z'_{n+1} \\ z'_{n+2} \\ z'_{n+3} \end{bmatrix}, A_2 = \begin{bmatrix} -\frac{2}{3} & 0 \\ -\frac{1}{3} & 0 \\ 1 & 0 \\ 2 & 0 \\ \frac{1}{h} & 0 \\ \frac{1}{h} & 1 \\ \frac{1}{h} & 0 \\ \frac{1}{h} & 0 \\ \frac{1}{h} & 0 \\ \frac{1}{h} & 0 \end{bmatrix}, R_1 = \begin{bmatrix} z_n \\ z'_n \end{bmatrix}, B_1 = \begin{bmatrix} -\frac{497}{87480}h^2 \\ -\frac{29}{10935}h^2 \\ -\frac{1}{15}h^2 \\ \frac{11}{120}h^2 \\ -\frac{32}{315}h \\ \frac{5687}{408240}h \\ \frac{131}{15515}h \\ \frac{11}{1008}h \\ -\frac{53}{315}h \\ \frac{3079}{5040}h \end{bmatrix}' R_2 = \left[g_n\right], B_2 =$$

$$\begin{bmatrix} -\frac{1}{15}h^2 & -\frac{53}{15120}h^2 & -\frac{31}{7290}h^2 & \frac{1}{11664}h^2 & -\frac{1}{153090}h^2 \\ -\frac{167}{4320}h^2 & -\frac{79}{1260}h^2 & -\frac{209}{29160}h^2 & \frac{1}{14580}h^2 & -\frac{11}{2449440}h^2 \\ \frac{81}{160}h^2 & -\frac{81}{140}h^2 & \frac{127}{120}h^2 & \frac{1}{12}h^2 & -\frac{3}{1120}h^2 \\ 0h^2 & \frac{243}{560}h^2 & \frac{13}{10}h^2 & \frac{89}{80}h^2 & \frac{13}{210}h^2 \\ -\frac{729}{2240}h^2 & -\frac{81}{1960}h^2 & -\frac{11}{336}h^2 & \frac{1}{840}h^2 & -\frac{13}{141120}h^2 \\ -\frac{13}{504}h^2 & -\frac{2231}{14112}h^2 & \frac{127}{34020}h^2 & -\frac{53}{1428840}h^2 & -\frac{131}{272160}h^2 \\ \frac{547}{4032}h^2 & \frac{839}{17640}h^2 & -\frac{2999}{136080}h^2 & \frac{5}{13608}h^2 & -\frac{269}{14430720}h^2 \\ \frac{27}{280}h^2 & \frac{2133}{7840}h^2 & \frac{17}{140}h^2 & -\frac{1}{1120}h^2 & \frac{1}{17640}h^2 \\ \frac{459}{448}h^2 & -\frac{621}{392}h^2 & \frac{3193}{1680}h^2 & \frac{281}{840}h^2 & -\frac{1021}{141120}h^2 \\ -\frac{729}{280}h^2 & \frac{35397}{7840}h^2 & -\frac{817}{420}h^2 & \frac{1097}{672}h^2 & \frac{1025}{3528}h^2 \end{bmatrix}, R_3 = \begin{bmatrix} g_{\frac{1}{3}} \\ g_{\frac{2}{3}} \\ g_1 \\ g_2 \\ g_3 \end{bmatrix} \tag{14}$$

Substituting Eq.(14) into Eq. (13) and multiply by the inverse of A_1 gives the hybrid block in the form:

$$IZ_1 = \bar{B}\,R_1 + \bar{D}_1\,R_2 + \bar{D}_2\,R_3 \qquad (15)$$

$$\begin{bmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{bmatrix} \begin{bmatrix} z_{n+\frac{1}{3}}\\ z_{n+\frac{2}{3}}\\ z_{n+1}\\ z_{n+2}\\ z_{n+3}\\ z'_{n+\frac{1}{3}}\\ z'_{n+\frac{2}{3}}\\ z'_{n+1}\\ z'_{n+2}\\ z'_{n+3} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3}h\\ 1 & \frac{2}{3}h\\ 1 & h\\ 1 & 2h\\ 1 & 3h\\ 0 & 1\\ 0 & 1\\ 0 & 1\\ 0 & 1\\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_n\\ z'_n \end{bmatrix} + \begin{bmatrix} \frac{17257}{612360}h^2\\ \frac{4981}{76545}h^2\\ \frac{32}{315}h^2\\ \frac{43}{315}h^2\\ \frac{11}{280}h^2\\ \frac{6737}{58320}h\\ \frac{389}{3645}h\\ \frac{9}{80}h\\ -\frac{1}{15}h\\ \frac{57}{80}h \end{bmatrix} \left[g_n\right] + \begin{bmatrix} \frac{281}{6780}h^2 & -\frac{2203}{105840}h^2 & \frac{2719}{408240}h^2 & -\frac{127}{408240}h^2 & \frac{829}{34292160}h^2\\ \frac{337}{1890}h^2 & -\frac{31}{882}h^2 & \frac{374}{25515}h^2 & -\frac{37}{51030}h^2 & \frac{61}{1071630}h^2\\ \frac{729}{2240}h^2 & \frac{281}{6780}h^2 & \frac{11}{336}h^2 & -\frac{1}{840}h^2 & \frac{13}{141120}h^2\\ \frac{81}{70}h^2 & -\frac{243}{490}h^2 & \frac{118}{105}h^2 & \frac{17}{210}h^2 & -\frac{11}{4410}h^2\\ \frac{2187}{2240}h^2 & \frac{2187}{3190}h^2 & \frac{783}{560}h^2 & \frac{621}{560}h^2 & \frac{195}{3136}h^2\\ \frac{826}{2880}h & -\frac{1177}{10080}h & \frac{709}{19440}h & -\frac{206}{178605}h & -\frac{1483}{3810240}h\\ \frac{83}{180}h & \frac{4}{45}h & \frac{13}{1215}h & -\frac{1}{1215}h & \frac{1}{14580}h\\ \frac{27}{64}h & \frac{351}{1120}h & \frac{37}{240}h & -\frac{1}{480}h & \frac{1}{6720}h\\ \frac{27}{20}h & -\frac{54}{35}h & \frac{29}{15}h & \frac{1}{3}h & -\frac{1}{140}h\\ -\frac{729}{320}h & \frac{729}{160}h & -\frac{153}{80}h & \frac{261}{160}h & \frac{93}{320}h \end{bmatrix} \begin{bmatrix} g_{\frac{1}{3}}\\ g_{\frac{2}{3}}\\ g_1\\ g_2\\ g_3 \end{bmatrix}$$

By putting Eq.(5) in Eq. (15) yield the proposed hybrid block schemes of the form:

$$y_{n+\frac{1}{3}} - f_{n+\frac{1}{3}} = y_n - f_n + \frac{1}{3}h\left(y'_n - f'_n\right) + \frac{17257}{612360}h^2 g_n + \frac{281}{6720}h^2 g_{n+\frac{1}{3}} - \frac{2203}{105840}h^2 g_{n+\frac{2}{3}}$$

$$+ \frac{2719}{408240}h^2 g_{n+1} - \frac{127}{408240}h^2 g_{n+2} + \frac{829}{34292160}h^2 g_{n+3}$$

$$y_{n+\frac{2}{3}} - f_{n+\frac{2}{3}} = y_n - f_n + \frac{2}{3}h\left(y'_n - f'_n\right) + \frac{4981}{76545}h^2 g_n + \frac{337}{1890}h^2 g_{n+\frac{1}{3}} - \frac{31}{882}h^2 g_{n+\frac{2}{3}} +$$

$$\frac{374}{25515}h^2 g_{n+1} - \frac{37}{51030}h^2 g_{n+2} + \frac{61}{1071630}h^2 g_{n+3}$$

$$y_{n+1} - f_{n+1} = y_n - f_n + h\left(y'_n - f'_n\right) + \frac{32}{315}h^2 g_n + \frac{729}{2240}h^2 g_{n+\frac{1}{3}} + \frac{81}{1960}h^2 g_{n+\frac{2}{3}} + \frac{11}{336}h^2 g_{n+1}$$

$$- \frac{1}{840}h^2 g_{n+2} + \frac{13}{141120}h^2 g_{n+3}$$

$$y_{n+2} - f_{n+2} = y_n - f_n + 2h\left(y'_n - f'_n\right) + \frac{43}{315}h^2 g_n + \frac{81}{70}h^2 g_{n+\frac{1}{3}} - \frac{243}{490}h^2 g_{n+\frac{2}{3}} + \frac{118}{105}h^2 g_{n+1}$$

$$+ \frac{17}{210}h^2 g_{n+2} - \frac{11}{4410}h^2 g_{n+3}$$

$$y_{n+3} - f_{n+3} = y_n - f_n + 3h\left(y'_n - f'_n\right) + \frac{111}{280}h^2 g_n + \frac{2187}{2240}h^2 g_{n+\frac{1}{3}} + \frac{2187}{3920}h^2 g_{n+\frac{2}{3}} + \frac{783}{560}h^2 g_{n+1}$$

$$+ \frac{621}{560}h^2 g_{n+2} + \frac{195}{3136}h^2 g_{n+3}$$

$$y'_{n+\frac{1}{3}} - f'_{n+\frac{1}{3}} = y'_n - f'_n + \frac{6737}{58320}h g_n + \frac{863}{2880}h g_{n+\frac{1}{3}} - \frac{1177}{10080}h g_{n+\frac{2}{3}} + \frac{709}{19440}h g_{n+1}$$

$$- \frac{206}{178605}h g_{n+2} - \frac{1483}{3810240}h g_{n+3}$$

$$y'_{n+\frac{2}{3}} - f'_{n+\frac{2}{3}} = y'_n - f'_n + \frac{389}{3645}h g_n + \frac{83}{180}h g_{n+\frac{1}{3}} + \frac{4}{45}h g_{n+\frac{2}{3}} + \frac{13}{1215}h g_{n+1}$$

$$- \frac{1}{1215}h g_{n+2} + \frac{1}{14580}h g_{n+3} \qquad (16)$$

$$y'_{n+1} - f'_{n+1} = y'_n - f'_n + \frac{9}{80}h g_n + \frac{27}{64}h g_{n+\frac{1}{3}} + \frac{351}{1120}h g_{n+\frac{2}{3}} + \frac{37}{240}h g_{n+1}$$

$$-\frac{1}{480}hg_{n+2}+\frac{1}{6720}hg_{n+3}$$

$$y'_{n+2}-f_{n+2}=y'_n-f_n-\frac{1}{15}hg_n+\frac{27}{20}hg_{n+\frac{1}{3}}-\frac{54}{35}hg_{n+\frac{2}{3}}+\frac{29}{15}hg_{n+1}+\frac{1}{3}hg_{n+2}$$

$$-\frac{1}{140}hg_{n+3}$$

$$y'_{n+3}-f_{n+3}=y'_n-f_n+\frac{57}{80}hg_n-\frac{729}{320}hg_{n+\frac{1}{3}}+\frac{729}{160}hg_{n+\frac{2}{3}}-\frac{153}{80}hg_{n+1}+\frac{261}{160}hg_{n+2}$$

$$+\frac{93}{320}hg_{n+3}$$

3.Analysis of the Hybrid Block Method

In this section, the analysis of the order, error constant, convergence and stability ofproposedschemeis carried out.

3.1.Order and error constant of the proposed method

Let the linear difference operator ℓ associated with the new method (16) be defined as

$$L[y(x;h)] = \sum_{j=0}^{k}\alpha_j y(x+jh) - h^2\sum_{j=0}^{k}(\beta_j y''(x+jh)) \quad (17)$$

Where$y(x)$ is an arbitrary test function continuously differential on [a, b]. Expanding $y(x+jh)$, $y'(x+jh$and $y''(x+jh$of (16) in Taylor series in the form:

$$\begin{bmatrix}
\sum_{j=0}^{\infty} \frac{\left(\frac{1}{3}h\right)^j}{j!}(z_n)^j - z_n - \frac{1}{3}hz_n' - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} g_n^{j+2}\left[\frac{1725}{612360} + \frac{281}{6720}\left(\frac{1}{3}\right) - \frac{2203}{1058040}\left(\frac{2}{3}\right) + \frac{2719}{408240}(1) - \frac{127}{408240}(2) + \frac{829}{3429460}(3)\right] \\
\sum_{j=0}^{\infty} \frac{\left(\frac{2}{3}h\right)^j}{j!}(z_n)^j - z_n - \frac{2}{3}hz_n' - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} g_n^{j+2}\left[\frac{4981}{76545} + \frac{337}{1890}\left(\frac{1}{3}\right) - \frac{31}{882}\left(\frac{2}{3}\right) + \frac{374}{25515}(1) - \frac{37}{51030}(2) + \frac{61}{1071630}(3)\right] \\
\sum_{j=0}^{\infty} \frac{(h)^j}{j!}(z_n)^j - z_n - hz_n' - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} g_n^{j+2}\left[\frac{32}{315} + \frac{729}{2240}\left(\frac{1}{3}\right) + \frac{81}{1960}\left(\frac{2}{3}\right) + \frac{11}{336}(1) - \frac{1}{840}(2) + \frac{13}{141120}(3)\right] \\
\sum_{j=0}^{\infty} \frac{(2h)^j}{j!}(z_n)^j - z_n - 2hz_n' - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} g_n^{j+2}\left[\frac{43}{315} + \frac{81}{70}\left(\frac{1}{3}\right) - \frac{243}{490}\left(\frac{2}{3}\right) + \frac{118}{105}(1) + \frac{17}{210}(2) - \frac{11}{4410}(3)\right] \\
\sum_{j=0}^{\infty} \frac{(3h)^j}{j!}(z_n)^j - z_n - 3hz_n' - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} g_n^{j+2}\left[\frac{111}{280} + \frac{2187}{2240}\left(\frac{1}{3}\right) + \frac{2187}{3920}\left(\frac{2}{3}\right) + \frac{783}{560}(1) + \frac{621}{660}(2) + \frac{195}{3136}(3)\right] \\
\sum_{j=0}^{\infty} \frac{\left(\frac{1}{3}\right)^j}{j!}(z_n)^j - z_n - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} g_n^{j+2}\left[\frac{6737}{58320} + \frac{863}{2880}\left(\frac{1}{3}\right) - \frac{1177}{10080}\left(\frac{2}{3}\right) + \frac{709}{19440}(1) - \frac{13}{7776}(2) + \frac{211}{1632960}(3)\right] \\
\sum_{j=0}^{\infty} \frac{\left(\frac{2}{3}\right)^j}{j!}(z_n)^j - z_n - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} g_n^{j+2}\left[\frac{389}{3645} + \frac{83}{180}\left(\frac{1}{3}\right) + \frac{4}{45}\left(\frac{2}{3}\right) + \frac{13}{125}(1) - \frac{1}{1215}(2) + \frac{1}{14580}(3)\right] \\
\sum_{j=0}^{\infty} \frac{(1)^j}{j!}(z_n)^j - z_n - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} g_n^{j+2}\left[\frac{9}{80} + \frac{27}{64}\left(\frac{1}{3}\right) + \frac{351}{1120}\left(\frac{2}{3}\right) + \frac{37}{240}(1) - \frac{1}{480}(2) + \frac{1}{6720}(3)\right] \\
\sum_{j=0}^{\infty} \frac{(2)^j}{j!}(z_n)^j - z_n - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} g_n^{j+2}\left[\frac{-1}{15} + \frac{27}{20}\left(\frac{1}{3}\right) - \frac{54}{35}\left(\frac{2}{3}\right) + \frac{29}{15}(1) + \frac{1}{3}(2) - \frac{1}{140}(3)\right] \\
\sum_{j=0}^{\infty} \frac{(3)^j}{j!}(z_n)^j - z_n - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} g_n^{j+2}\left[\frac{57}{80} - \frac{729}{160}\left(\frac{1}{3}\right) + \frac{729}{160}\left(\frac{2}{3}\right) - \frac{153}{80}(1) + \frac{261}{160}(2) + \frac{93}{320}(3)\right]
\end{bmatrix} \quad (18)$$

If we assume that $y(x)$ has many higher derivatives and collecting the terms, we have:

$$\ell[y(x);h] = \bar{c}_0 y(x) + \bar{c}_1 h y'(x) + \bar{c}_2 h^2 y''(x) + \cdots + \bar{c}_{p+2} h^{p+2} y^{(p+2)}(x) \quad (19)$$

According to Aruchuman and Sulaiman(2010), the proposed scheme has order $p$if,$\bar{c}_0 = \bar{c}_1 = \bar{c}_{2=} \ldots = \bar{c}_{p+1} = 0, \bar{c}_{p+2} \neq 0$

The proposed method is of order $p = [7,7,7,7,7,7,7,7,7,7]^T$, with error constant

C_9

$$= \left[\frac{286}{793618560}, \frac{107}{12400290}, \frac{1}{72576}, \frac{1}{5670}, \frac{11}{13440}, \frac{281}{14696640}, \frac{281}{2755620}, \frac{11}{544320}, \frac{17}{34020}, \frac{13}{4032}\right]^T$$

3.2. Consistency of the method

According to Areo and Omojola (2015), the hybrid block method is said to be consistent if it has an order more than or equal to onei.e $(p \geq 1)$. Since the Eq. (16) is of order $p = 7$, therefore, the proposed hybrid block method is consistent

.2.3. Zero-stability of the proposed method

The linear multistep hybrid block method is said to be zero-stable as $h \to 0$, if the roots of the first characteristics polynomial defined by$\rho(z) = det\left[\sum_{j=o}^{k} A^{(i)} Z^{(K-i)}\right]$ satisfies $|z| \leq 1$ and every root of$|z| = 1$ has multiplicity not exceeding the order of the differential equation. Awoyemi *et.al.*(2011)

$$A^0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ o & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, A' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ o & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We have$\rho(z) = z^8(z-1) = 0, z = 0.0.0.0.0.0.0,0,1$

Since $|z| = 1$, therefore. the proposed method is zero-stable

2.4. Convergence of the proposed method The necessary and sufficient condition for a linear multistep hybrid block method to be convergent is to be consistent and zero stable. Since, our scheme satisfies the two conditions, hence the Eq. (9) is convergent.

4. Implementation of method In this section, we implement the proposed hybrid block method on two considered problems of second kind VIEs.

Problem 4.1

Consider the second kind linear volterra integral equation

$X(t) = t^2 + \int_0^x (t-s)x_1(s)ds$

With exact solution$X(t) = 2cosht - 2, h = 0.1$

Source: Muturi *et al*. (2014)

Converting the nonlinear VIE to a second order ODE, we obtained $x(t) = t^2 + \int_0^t (t-s)\, x_1(s)ds$ $x'(t) = 2t + \int_0^t x_1(s)\, ds$ $x''(t) = 2 + x_1(s)$

Then, the second order ODE is given as $z'' = 0z_0' + z_0 + 2$

Where$z_0 = 0,\ z' = 0.$

Table .1 Showing the exact solution and computed results from the propose methods for problem 1

X	Exact	Numerical Result	Error in Muturi *et al*. (2014)	Error in Proposed Method
0.1	0.01000833611160719800	0.01001667222375875757	1.0 E-05	2.7763 E-13
0.2	0.04013351123815169260	0.04013351123459477332	3.0 E-05	3.5569 E-12
0.3	0.09067702825772097000	0.09067702827425125113	8.0 E-05	1.6530 E-11
0.4	0.16214474367690961860	0.16214474375907952666	1.40 E-04	8.2169 E-11
0.5	0.25525193041276157040	0.25525193055694029498	2.20 E-04	1.4417 E-10
0.6	0.37093043648453540760	0.37093043671821427031	3.20 E-04	2.33678 E-10
0.7	0.51033801126188603640	0.51033801163676863682	4.40 E-04	3.74882 E-10
0.8	0.67486989260968919600	0.67486989312433225989	5.90E-04	5.14643 E-10
0.9	0.86617277089754877560	0.86617277158760432182	7.70E-04	6.9005 E-10
1.0	1.08616126963048755700	1.08616127056068089180	9.80 E-04	9.3019 E-10

Problem 2 **Consider the second kind linear volterra integral equation**

$U(x) = 1 + x + \int_0^x (x - t)U(t)dt$ With exact solution:

$$U(x) = e^x, \qquad h = 0.1$$

Source: Shoukralla and Ahmed (2020) Converting the VIE to a second order ODE, we obtained $u(x)1 + x + \int_0^x (x - t)\, u(t)dt$

$u'(x) = 1 + \int_0^x u(t)\,dt u''(x) = u(x)$, The second order ODE is then given as$z'' = 0z' + z$

where$z_0 = 0, z' = 0$.**Table 2 Showing the exact solution and computed results from the propose methods for problem 2**

X	Exact	Numerical Result	Error in Shoukralla and Ahmed (2020)	Error in Proposed Method
0.1	1.1051709180756476248	1.10517091807580048840	1.4089 E-09	1.5286 E-13
0.2	1.2214027581601698339	1.22140275815817059020	9.1493 E-08	1.9992 E-12
0.3	1.3498588075760031040	1.34985880758546021050	1.0576 E-05	9.4571 E-12
0.4	1.4918246976412703178	1.49182469768812457810	6.0309 E-06	4.6854 E-11
0.5	1.6487212707001281468	1.64872127078173504370	2.3354 E-05	8.1607 E-11
0.6	1.8221188003905089749	1.82211880052608176310	7.08004 E-05	1.3557 E-10
0.7	2.0137527074704765216	2.01375270769626071760	1.8129E-04	2.2578 E-10
0.8	2.2255409284924676046	2.22554092880652021370	4.1026 E-04	3.1405 E-10
0.9	2.4596031111569496638	2.45960311158724651560	8.4486 E-04	4.3029 E-10
1.0	2.7182818284590452354	2.69286592330700708760	1.6151 E-03	2.5416 E-02

Discussion and Conclusion

In this research, Continuous Linear Multistep Hybrid Block Method was proposedfor the solution of integral equation of second kind. We investigated the property of this proposed method in terms of order, error constant, consistency, zero-stability and convergence analysis. The scheme was also used to solve numerically two problems of volterra integral equation of second kind and the results were compared with[9] and [10].From the Table 1 and 2, we discovered that, the proposed multistep hybrid block method were capable of
Handling the second kind VIEs. The results obtained from Table 1 and 2 indicated that our proposed methodsare considerably much more accurate than existing numerical method. All computation and program were carried out with the aid of MAPPLE software.

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