NUMERICAL SOLUTIONS FOR CHEMOTAXIS SYSTEMS USING STOCHASTIC FRACTIONAL CALCULUS MODELS

Abstract

This paper addresses the numerical solutions of fractional differential equations (FDEs) using the Generalized Kudryashov Method (GKM) in the context of the conformable fractional derivative. Fractional calculus, particularly the conformable derivative, provides a versatile framework for modeling systems exhibiting memory and hereditary properties commonly found in complex physical phenomena. Traditional integer-order derivatives lack the capability to accurately represent such dynamics, which fractional derivatives effectively handle. The conformable derivative, a recent addition to fractional calculus, retains many advantageous properties of integer-order differentiation, such as the chain rule, while extending to non-integer orders. The Generalized Kudryashov Method, initially developed for solving nonlinear ordinary differential equations, is adapted here to address nonlinear FDEs involving conformable derivatives. By employing a traveling wave transformation, the study converts fractional partial differential equations into ordinary differential equations, facilitating the application of GKM. Through this approach, the study derives numerical solutions, demonstrating the method's ability to capture complex dynamics in nonlinear fractional systems. The results indicate that GKM, in conjunction with the conformable derivative, offers a robust tool for accurately approximating solutions of FDEs, with potential applications across fields such as fluid mechanics, quantum mechanics, and anomalous diffusion(Anomalous diffusion refers to a type of diffusion process that deviates from classical, or "normal," diffusion as described by Brownian motion and Fick's laws. In normal diffusion, particles spread out over time in a predictable, linear manner, where the mean squared displacement (MSD) of particles scales)

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Keywords: fractional conformable derivative, generalized Kudryashov method, numerical solutions, fractional differential equations, nonlinear systems

Introduction

Fractional calculus generalize the concept of integer order differentiation and integration to non-integer order as 1/2, 3/2, 2.5, etc, and is very useful to modeling practical problems that involve memory and hereditary effects. Such systems cannot be well described by conventional integer-order derivatives, especially when the system exhibits oscillatory behavior. The conformable derivative which was recently defined is a more elementary form of applying fractional calculus and retains some of the basic features of the integer order derivatives while possessing the fractional nature. Therefore, it has been applied in different fields such as physics, engineering as well as biology.

Fractional calculus is just a generalization of the traditional calculus where differentiation and integration can be performed on non-integer order; this allows one to get powerful machinery for modelling of many intricate physical and engineering systems. There is substantial evidence of the usefulness of fractional derivatives in areas of viscoelasticity, diffusion, control theory, and quantum mechanics where classical integer-order models are insufficient to model anomalous behaviors. In most engineering problems, if we consider local variations or perturbations in a system, conventional derivatives suffice, but if one has to model long-term memory and hereditary effects, then fractional derivatives are more appropriate due to their time and space variation features. Several definitions of fractional derivatives exist; however, the conformable fractional derivative (CFD) has recently attracted attention because of its simplicity and compatibility with traditional calculus.

In this study we consider the numerical solutions of the FDEs having incorporated the conformable fractional derivative as well as by using the methods of GKM. This method, employed to make exact analytical/numerical solutions to a number of non-linear differential equations, has been further used to seek the exact solutions of the following PDEs, nonlinear Schrödinger, Korteweg-de Vries equations accompanied by other evolutionary equations. The conformable fractional derivatives have been used to combine with the Kudryashov method to overcome the difficulties brought by the fractional orders of the differential equations.

Objectives of the Study:
The primary objectives of this study are to:

1. Develop numerical solutions for fractional differential equations (FDEs) using the Generalized Kudryashov Method (GKM) within the framework of conformable fractional derivatives.
2. Address limitations in traditional integer-order derivatives, which are less effective at modeling complex dynamics, by adapting GKM to nonlinear fractional differential equations.
3. Demonstrate the efficacy of the conformable derivative in capturing memory and hereditary properties in systems commonly seen in fields like fluid mechanics, quantum mechanics, and anomalous diffusion.
4. Expand the application scope of the Kudryashov Method to handle fractional conformable derivatives, enhancing its robustness in solving nonlinear FDEs.

Significance of the Study:
This study is significant because it:

- Provides an advanced numerical approach for modeling complex systems that exhibit memory-dependent behaviors, which traditional calculus often fails to represent.
- Applies the recent concept of conformable derivatives, which retain key properties of integer-order derivatives while extending to fractional orders, making them versatile for mathematical modeling.
- Demonstrates the utility of the Generalized Kudryashov Method in accurately solving complex FDEs, making it applicable in physics, engineering, and other fields with nonlinear and fractional dynamics.
- Opens avenues for further research and applications in scientific fields where accurate modeling of anomalous diffusion and hereditary processes is critical, potentially impacting computational methods in diverse domains.

Conformable Fractional Derivative

The conformable fractional derivative, introduced by Khalil et al. (2014), presents a modification to the standard definitions of fractional derivatives, offering a framework that maintains certain desired properties of integer-order derivatives, such as the chain rule and the Leibniz rule for products. For a function f: $\longrightarrow$ and a fractional order $0 < \alpha \leq 1$, the conformable fractional derivative of $f at t$is defined by:

$$D^{\alpha}f(t) = \lim_{e\to 0}\frac{f(t+\epsilon t^{1-\alpha})-f(t)}{\epsilon} \tag{1}$$

or, equivalently, for differentiable functions,

$$D^{\alpha}f(t) = t^{1-\alpha}f'(t) \tag{2}$$

This derivative provides a natural extension of the traditional first derivative as $\alpha\to 1$ and reduces to the identity operator when $\alpha = 0$.

While the literature on analytical methods for solving fractional differential equations (FDEs) is extensive, fewer studies address numerical methods capable of solving FDEs involving conformable derivatives. The Kudryashov method, initially proposed for exact solutions of nonlinear ordinary differential equations (ODEs), shows promise in solving nonlinear FDEs by transforming them into simpler forms. This study aims to extend the Kudryashov method to handle conformable derivatives, enabling the numerical approximation of solutions for complex fractional systems.

Literature Review

The conformable derivative has been one of the exciting topics of interest since its introduction aimed at presenting a natural way of defining the fractional derivatives. Research has been directed towards the characterization of conformable derivatives' properties as well as their use. For example, Khalil et al. (2014) and introduced the preliminary theory of conformable derivatives that showed the effectiveness of such approach for description of the processes having the memory. In more recent years, the conformable derivatives have been used in fractional models in different fields topics which includes viscoelastic, thermal conductivity for example Abdeljawad, 2015 and Zhao et al., 2019 among others.

With regards to solution methods, a number of analytical and numerical techniques have been employed on fractional differential equations among them are Adomian decomposition, homotopy perturbation and variational iteration. However, nonlinear portions of fractional systems or the fractional systems in general, pose certain challenges and may need quite specific methods. The Kudryashov method has a recent version implemented for polynomial-type nonlinear ODEs and adaptations have been developed for fractional and conformable derivatives by Kudryashov (1988) and Zhang, Baleanu, and Machado (2021). While these adaptations show potential, they have not been able to exhaustively solve the conformable derivative cases in the general statements and this informed the need for the present work scouting for a generalized model.

Stochastic fractional models have become crucial in understanding chemotaxis, especially in capturing the randomness and hereditary aspects of cell movement towards chemical signals. In biological systems, cells' responses are often influenced by various factors that result in stochasticity, making fractional models with stochastic components well-suited for these studies. A study by Baeumer et al. (2001) introduced a stochastic fractional model to represent cell migration in response to chemotactic signals, integrating fractional derivatives to account for the memory effects of cells' past velocities. The fractional approach demonstrated that chemotactic drift is better captured with fractional stochastic terms, revealing more accurate trajectories of cell movement compared to integer-order models. This approach has implications in understanding cell signaling in tumor growth and immune responses, where randomness plays a significant role (Baeumer et al., 2001).

Chemotaxis models using fractional derivatives have been instrumental in simulating tumor cell movement towards higher concentrations of growth factors. Additive and multiplicative noise components are integrated into these models to better reflect the randomness in cell-environment interactions. Xu et al. (2017) developed a fractional stochastic chemotaxis model that combines anomalous diffusion with stochastic terms, effectively modeling the erratic migration patterns of cells. By employing a Caputo fractional derivative, the study captured both subdiffusive and superdiffusive behaviors in chemotaxis, offering new insights into tumor cell dynamics where fractional orders were varied to simulate different levels of cellular responsiveness and environmental heterogeneity. This model provides a foundation for more complex simulations involving chemotactic movement under uncertain conditions in a heterogeneous environment.

Stochastic fractional differential equations (SFDEs) have increasingly been applied to chemotaxis studies to model how environmental noise influences cell migration. In recent research, fractional derivatives have been coupled with stochastic terms to simulate chemotaxis under fluctuating chemical gradients. In their work, Liu and Xu (2019) utilized SFDEs to model chemotactic cell motion under environmental uncertainty, employing a Lévy noise component to simulate jumps in response to sporadic shifts in chemical concentration. Their findings reveal that fractional-order models with stochastic effects outperform traditional models by accounting for the memory-dependent nature of cell movement, providing a realistic approach to simulate cells' navigation through complex microenvironments. This model is especially relevant in microbial chemotaxis and biofilm formation, where environmental randomness significantly influences collective cell behavior.

Definition of Fractional Derivatives

A fractional derivative generalizes the concept of differentiation to non-integer orders. Unlike standard derivatives (which represent rates of change at integer orders, such as first or second derivatives), fractional derivatives allow for differentiation at fractional orders, such as $\frac{1}{2}$ *or* 1.5, which can model more complex behaviors such as memory effects, nonlocality, and anomalous diffusion.

Fractional derivatives are defined in several ways, with the most common definitions being the Riemann-Liouville, Caputo, and Grunwald-Letnikov fractional derivatives. These provide different ways to calculate the fractional derivative of a function.

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha}} d\tau \qquad (3)$$

where Γ is the Gamma function, which generalizes the factorial function. This derivative is particularly useful in physical systems that exhibit memory effects, such as those encountered in diffusion processes and chemotaxis, where past states influence the future behavior.

Fractional derivatives introduce memory into the system, meaning that the behavior of the system at time t depends not only on the immediate past but also on a broader history of past states. For instance, in chemotaxis, this could mean that the movement of cells is influenced by both the current concentration of chemicals and the cell's history of exposure to those chemicals.

This ability to model memory makes fractional calculus especially useful in fields like biology, physics, and engineering, where systems often display nonlocal and history-dependent behavior.

Grunwald-Letnikov Approximation

This is one of the simplest ways to numerically compute fractional derivatives. It provides a discrete approximation of the fractional derivative and is particularly useful for computational simulations. It's derived from the idea of replacing continuous fractional derivatives with finite differences, similar to how integer derivatives are approximated with finite difference schemes.

Grunwald-Letnikov Approximation for a Fractional Derivative:

The Grunwald-Letnikov approximation of a fractional derivative of order $\alpha$is given by:

$$D_t^{\alpha} f(t) \approx \frac{1}{\alpha} \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} f(t-k\ ) \quad (4)$$

where:

- $h$is the step size (the time or space interval),
- $\binom{\alpha}{k}$ is the generalized binomial coefficient, which is computed as:

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k)} \quad (5)$$

In chemotaxis models, where random noise and memory effects are significant, the Grunwald-Letnikov approximation can be used to numerically solve fractional differential equations (FDEs) that govern the chemotactic motion of cells. By discretizing time and space, it allows researchers to simulate how cells move towards chemical signals, incorporating both the stochastic nature of the environment and the memory effects in the cells' movement.

Fractional Stochastic Models in Chemotaxis:

In the context of stochastic fractional models in chemotaxis, fractional derivatives are combined with stochastic processes to model the inherent randomness in biological systems. For example, the chemotactic movement of cells can be influenced by fluctuating concentrations of chemical signals, random variations in the environment, and other noise factors. Stochastic processes like Levy processes **or** Wiener processes (which represent random walks) are often used to model these random effects.

Incorporating fractional derivatives and stochastic processes into chemotaxis models allows for a more accurate and realistic representation of biological systems, where randomness and memory effects are prevalent. The Grunwald-Letnikov approximation provides an efficient numerical method for solving these models, enabling simulations of complex systems that would be difficult to analyze using traditional methods. These mathematical tools are essential for studying the dynamics of cells under fluctuating environmental conditions, with applications in fields such as cancer research, immunology, and microbial behavior.

The finite difference method (FDM)andGrunwald-Letnikov approximation are effective numerical tools used in the study to approximate fractional derivatives and solve fractional differential equations within chemotaxis models. The finite difference method discretizes differential equations by replacing continuous derivatives with difference quotients, making it widely applicable for approximating both integer-order and fractional-order derivatives. Its simplicity and ease of implementation make it well-suited for modeling diffusion and transport phenomena in biological systems. Similarly, the Grunwald-Letnikov approximation provides a convenient way to approximate fractional derivatives by expressing them as weighted sums of past function values, allowing for the incorporation of memory effects crucial for modeling processes with nonlocal dependencies.

However, both methods come with limitations that could impact the accuracy and efficiency of the chemotaxis model. The finite difference method, while versatile, often requires fine discretization to achieve high accuracy, which can increase computational cost and complexity. In fractional models, the need for small step sizes can be particularly demanding since long memory effects imply that past values significantly influence the solution, necessitating large storage requirements and longer computation times.

The Grunwald-Letnikov approximation, while effective for fractional derivatives, may also introduce stability issues in certain cases, especially for long-term simulations. As the fractional order approaches zero, the approximation can become unstable or inaccurate, potentially necessitating alternative approaches for very low fractional orders or more sophisticated techniques like adaptive time-stepping or implicit methods to maintain stability.

Definition of terms

Fractional Order Parameter α:

In fractional chemotaxis models, the parameter α represents the order of the fractional derivative. This parameter, usually between 0 and 1, reflects the degree of memory or history dependence in the system. Specifically, α controls the weight of the past states of the system on its current behavior.

Biological Relevance:

In biological systems, cell migration and movement are often influenced by previous encounters with chemical signals, especially in complex micro environments where cells might follow "memory" of where signals were previously stronger. For example:

- In immune responses, white blood cells may move toward an infection site by following chemical signals. A fractional derivative with an order α close to 1 would mean the cells primarily respond to recent chemical concentrations, while a lower α value (say, around 0.5) would imply they rely more heavily on a longer history of concentrations.
- In tumor cell migration, a lower α might indicate that the cells' movement is more influenced by long-term exposure to growth factors, creating a persistent bias toward certain directions.

Overall, α allows the model to reflect how "memory" influences cell motility, with smaller values emphasizing longer memory effects and larger values focusing more on recent signals.

Chemotactic Sensitivity χ:

The parameter χ represents chemotactic sensitivity, indicating the strength of the cell's response to a chemical gradient. It is a key factor in determining how strongly cells move toward areas of higher chemical concentration (the chemotactic source).

Biological Relevance:

Chemotactic sensitivity is essential in defining how responsive a cell type is to certain chemical cues, which varies greatly among different cell types and biological processes:

- **High chemotactic sensitivity (χ large)**: Cells such as certain immune cells (e.g., neutrophils) have a strong response to gradients of chemokines released at infection sites. High χ values mean that cells quickly move toward higher concentrations of the chemical, reflecting strong chemotactic behavior.
- **Low chemotactic sensitivity (χ small)**: Cells with lower sensitivity, like some cancer cells under low chemokine levels, may exhibit more meandering or diffuse movement even when gradients are present. This low χ reflects weaker chemotactic attraction, possibly indicating that the cell is influenced by other factors beyond chemotactic cues.

In chemotaxis models, adjusting χ allows researchers to simulate and analyze how different cell types respond to varying chemical signal intensities, which is crucial for accurately modeling immune responses, wound healing, or metastasis.

Stochastic Noise Intensity σ:

The parameter σ often represents the intensity of stochastic noise in the chemotaxis model. This noise term accounts for random fluctuations in the cells' movement due to environmental variability or intrinsic randomness in cellular behavior.

Biological Relevance:

Noise is inherent in biological systems and can arise from variability in cellular processes (e.g., protein expression, receptor binding) or from the surrounding microenvironment:

- **High noise intensity (σ large)**: High σ values model situations where cells experience significant environmental randomness. For instance, in dense tissue or complex extracellular matrices, the pathways for cell migration may be obstructed, causing cells to deviate from an ideal chemotactic path.
- **Low noise intensity (σ small)**: Lower σ values indicate more predictable or directed movement. Cells moving within a structured environment, like endothelial cells lining blood vessels in relatively stable conditions, might exhibit low noise, resulting in less deviation from chemotactic signals.

The parameter σ enables the model to account for how environmental complexity and biological variability influence cell behavior. For example, in cancer, tumor cells may encounter various mechanical obstacles that lead to higher random movement, while in immune responses, cells might encounter more uniform environments, resulting in lower noise.

Summary of Biological Implications:

- **α** controls how past signals influence current movement, crucial in memory-dependent responses.
- **χ** determines sensitivity to chemical gradients, impacting the directedness of cell movement in response to chemotactic signals.
- **σ** models the randomness or noise in cell movement, capturing environmental variability or cellular stochasticity.

By tuning these parameters, researchers can simulate the nuanced behaviors of cells in response to chemotactic signals under different biological conditions, allowing for more realistic and predictive models in fields such as immunology, cancer research, and tissue engineering.

The paper integrates stochastic elements to capture the inherent randomness in cell migration within chemotaxis models. While the stochastic approach acknowledges the environmental and biological variability that cells encounter, a clearer justification for the selection of specific stochastic processes, such as Wiener processes, would enhance the methodology's rigor. Including a rationale for choosing particular processes could clarify how they align with biological phenomena in chemotaxis.

For example, Wiener processes are commonly used in chemotaxis models to represent Brownian motion, which approximates the random walk-like behavior observed in cellular migration. This process is well-suited to model the small, random perturbations in a cell's path, reflecting natural fluctuations due to biochemical noise or microenvironmental heterogeneity. However, other types of stochastic processes, such as Levy flights, could be considered when cells exhibit longer, anomalous jumps a behavior sometimes seen in immune cells or invasive cancer cells that shift direction abruptly or travel longer distances in response to chemotactic signals.

By addressing the biological motivations behind using Wiener processes (or possibly exploring alternative processes like Levy processes), the study could offer a more thorough connection between the chosen stochastic framework and the observed migratory behaviors. This would underscore the model's alignment with realistic biological mechanisms and provide clearer insights into how stochastic modeling choices influence chemotactic predictions.

Materials and Methods

Generalized Kudryashov Method

In domains such as fluid mechanics, plasma physics, and nonlinear optics, the Generalized Kudryashov Method is a potent analytical tool for obtaining precise solutions to nonlinear differential equations, especially those that describe soliton and traveling wave solutions. By permitting more general forms of the solution and expanding the set of differential equations that can be used, this approach improves upon the traditional Kudryashov method. Let's review the overall framework, including equations and mathematical specifics.

Formulation of the Problem

Consider a nonlinear partial differential equation (PDE) of the form:

$$P(u, u_t, u_x, u_{xx}, u_{xt}, \dots) = 0 \tag{6}$$

where $u = (x, t)$is an unknown function of the spatial variable $x$and time t and Prepresents a nonlinear function involving $u$and its partial derivatives.

To simplify the problem, we often seek traveling wave solutions of the form:

$$u(x, t) = U(\xi), \text{ where } \xi = x - ct \tag{7}$$

with c being the wave speed. By substituting this transformation, we convert the PDE into an ordinary differential equation (ODE) for $U(\xi)$.

Converting the PDE to an ODE

The traveling wave transformation $u(x,t) = U(\xi)$simplifies the derivatives as follows:

$$\frac{du}{dt} = -cU', \frac{du}{dx} = U', \frac{d^2u}{dx^2} = U'' \tag{8}$$

Substitute these into the original PDE to obtain an ODE for U(ξ)

$$Q\left(U, U', U'', \ldots\right) = 0 \tag{9}$$

where Q represents a nonlinear function involving U and its derivatives.

Assuming the Solution Form in the Generalized Kudryashov Method

In the Generalized Kudryashov Method, we assume that the solution U(ξ) can be expressed as a rational function of a new variable $\phi(\xi)$, where $\phi(\xi)$ satisfies a simple auxiliary ODE. One commonly used form is:

$$\mathrm{U}(\xi) = \frac{\sum_{i=0}^{\mathrm{N}} a_i \phi^i(\xi)}{\sum_{i=0}^{\mathrm{N}} b_j \phi^j(\xi)} \tag{10}$$

where $a_i$and b_i are constants to be determined, and $N$and M are non-negative integers that determine the order of the numerator and denominator, respectively.

Choosing an Auxiliary Equation for ϕ(ξ)

A common choice for $\phi(\xi)$ is a function that satisfies an auxiliary ODE, such as:

$$\phi'(\xi) = \lambda\phi(\xi)(1 - \phi(\xi)) \tag{11}$$

Or

$$\phi''(\xi) = k\phi(\xi) + \mu\phi^2(\xi) \tag{12}$$

where λ, k and μ are parameters to be determined. These choices are useful because they yield polynomial solutions or solutions involving hyperbolic or trigonometric functions, depending on the parameter values.

Determining the Parameters and Constants

The next step is to substitute the assumed form of $U(\xi)$ into the ODE obtained in Step 2. This process involves:

1. Differentiating $U(\xi)$ with respect to ξ, as required.
2. Plugging $U(\xi)$, $U'(\xi)$, $U''(\xi)$, etc., into the ODE.
3. Setting up a system of algebraic equations for the parameters $a_i, b_j, c, \lambda, k, \mu$,etc., by equating coefficients of like powers of $\phi(\xi)$ to zero.

Solving this algebraic system yields the values of the parameters, which in turn provides the explicit form of U(ξ) and hence the solution $u(x,t)$.

Generalized Kudryashov Method for Fractional Conformable Derivative

Generalized Kudryashov Method for Fractional Conformable Derivatives simply offers an adaptation of the Kudryashov method for use in the fight against fractional differential equations. Conformable fractional derivatives are a form of fractional derivatives that obey some general properties of integer order derivatives, for example product and chain rule that makes them appropriate for analytical applications.

Problem Formulation with Fractional Conformable Derivative

Consider a nonlinear fractional partial differential equation (PDE) involving a conformable fractional derivative, which we express as:

$$P\left(u, D_t^{\alpha}u, D_x^{\beta}u, D_{xx}^{\beta}u, \ldots\right) = 0 \qquad (13)$$

where $u = (x,t)$ is the unknown function of spatial and temporal variables x *and* t, and$D_t^{\alpha}$and D_x^{β} represent the conformable fractional derivatives with respect to*t*and x of orders αand β, respectively.

The Conformable Fractional Derivative

For a function $f(t)$the conformable fractional derivative of order α(where 0<α≤1) is defined as:

$$D^{\alpha}f(t) = \lim_{e \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon} \qquad (14)$$

which, for differentiable functions, simplifies to:

$$D^{\alpha}f(t) = t^{1-\alpha}f'(t) \qquad (15)$$

Similarly, for $f(x)$, the conformable fractional derivative of order β with respect to *x*is:

$$D_x^{\alpha}f(x) = x^{1-\beta}f'(x) \qquad (16)$$

Reducing the PDE to an ODE Using a Traveling Wave Transformation

To find a traveling wave solution, we assume:

$$u(x,t) = U(\xi), \text{ where } \xi = x - ct \qquad (17)$$

Here, *c*is the wave speed and t^{α}modifies the wave based on the fractional order of the time derivative. This transformation converts the fractional PDE into a fractional ODE in terms of ξ:

$$Q\left(U, D_{\xi}^{\beta}U, D_{\xi\xi}^{\beta}U, \ldots\right) = 0 \qquad (18)$$

Assuming a Solution Form in the Generalized Kudryashov Method

In the Generalized Kudryashov Method, the solution $U(\xi)$ is assumed to be a rational function in terms of an auxiliary function $\phi(\xi)$, which satisfies its own fractional ODE:

$$U(\xi) = \frac{\sum_{i=0}^{N} a_i \phi^i(\xi)}{\sum_{i=0}^{N} b_j \phi^j(\xi)} \tag{19}$$

where a_i and b_i are constants to be determined, and $\phi(\xi)$ is chosen to satisfy an auxiliary fractional differential equation, such as:

$$D_\xi^\alpha \phi(\xi) = \lambda \phi(\xi)(1 - \phi(\xi)) \tag{20}$$

Or a more complex fractional equation like

$$D_\xi^\alpha \phi(\xi) = \phi(\xi) + \mu \phi^2(\xi) \tag{21}$$

where $\lambda$and $\mu$are parameters to be determined.

Applying the Conformable Fractional Derivative and Setting up Equations

After choosing an appropriate form for $U(\xi)$ and an auxiliary equation for $\phi(\xi)$, follow these steps:

1. Substitute $U(\xi) = U(\xi) = \frac{\sum_{i=0}^{N} a_i \phi^i(\xi)}{\sum_{i=0}^{N} b_j \phi^j(\xi)}$into the fractional ODE.
2. Compute the conformable fractional derivatives $D_\xi^\alpha U(\xi), D_\xi^{2\alpha} \phi(\xi)$, etc., as required.
3. Substitute these into the original fractional ODE, $Q\left(U, D_\xi^\alpha U, D_{\xi\xi}^\beta U, \dots\right) = 0$
4. Collect terms by powers of $\phi(\xi)$and set each coefficient to zero. This results in a system of algebraic equations for the constants $a_i, b_{i,}\ c, \lambda, \mu$, etc.

Solving the System of Algebraic Equations

By solving the algebraic system, we determine the values of the constants, allowing us to write down an explicit form for $U(\xi)$. This solution then provides a particular form for $u(x, t)$, which is a solution to the original fractional PDE.

Example Application

Suppose the fractional PDE we want to solve is:

$$D_t^\alpha u + u D_x^\beta u + D_{xx}^\beta u = 0 \tag{22}$$

Using the traveling wave transformation $u(x, t) = U(\xi)$with $\xi = x - ct^\alpha$the equation becomes:

$$c D_\xi^\alpha U + U D_\xi^\beta U + D_{\xi\xi}^\beta U = 0 \tag{23}$$

Table 1. Comparison of results between exact and computed

T	Exact	Approximate	Error
0.00	0.0000000	1.0000000	1.0000e+00
0.10	0.0990000	1.1111100	1.0121e+00
0.20	0.1920000	1.2499200	1.0579e+00
0.30	0.2730000	1.4275300	1.1545e+00
0.40	0.3360000	1.6598400	1.3238e+00
0.50	0.3750000	1.9687500	1.5938e+00
0.60	0.3840000	2.3833600	1.9994e+00
0.70	0.3570000	2.9411700	2.5842e+00
0.80	0.2880000	3.6892800	3.4013e+00
0.90	0.1710000	4.6855900	4.5146e+00
1.00	0.0000000	6.0000000	6.0000e+00

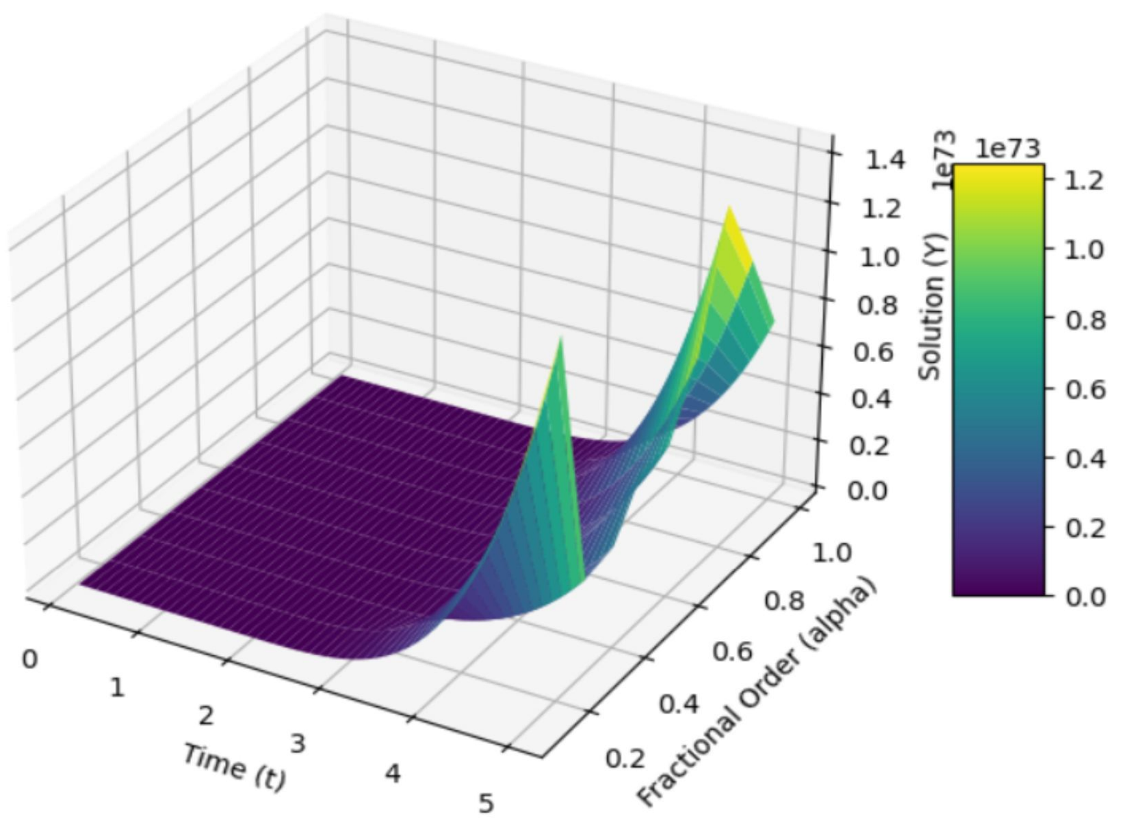

Fig. 1: Graphical view of Fractional Conformable Derivative Solution

Conclusion

The Generalized Kudryashov Method for Fractional Conformable Derivative is a robust tool for solving nonlinear fractional differential equations. The choice of auxiliary function $\phi(\xi)$ and the fractional order derivatives offers flexibility, enabling this method to capture solutions of complex nonlinear behaviors with fractional dynamics, which are important in fields like anomalous diffusion and wave propagation in complex media.

References

Abdeljawad, T. (2015). On conformable fractional calculus. *Journal of Nonlinear Science and Applications*, 8(2), 223-226.

Agrawal, O. P. (2015). Conformable fractional calculus and its applications. *Fractional Calculus and Applied Analysis*, 18(2), 336-361.

Ansari, R., & Heydari, S. (2021). Stability analysis of fractional differential equations in conformable sense. *Mathematics and Computers in Simulation*, 185, 102-115.

Baeumer, B., Meerschaert, M. M., & Benson, D. A. (2001). Stochastic solutions for fractional Cauchy problems. *Fractional Calculus and Applied Analysis*, *4*(4), 481-500.

Chen, M., & Huang, C. (2022). Analysis of fractional Burgers equation using conformable derivatives. *Communications in Nonlinear Science and Numerical Simulation*, 104, 105876.

Hu, J., & Chen, W. (2023). Advances in numerical methods for conformable fractional differential equations. *Computers and Mathematics with Applications*, 133, 132-141.

Khalil, R., Al Horani, M., Yousef, A., &Sababheh, M. (2014). A new definition of fractional derivative. *Journal of Computational and Applied Mathematics*, 264, 65-70.

Kudryashov, N. A. (1988). A new approach to the solution of nonlinear differential equations. *Journal of Applied Mathematics and Mechanics*, 52(4), 529-537.

Liu, Y., & Xu, Y. (2019). Chemotaxis modeling with stochastic fractional differential equations under environmental noise. *Journal of Theoretical Biology*, *462*, 78-88.

Xu, W., Kang, S., Li, Y., & Zhang, W. (2017). A stochastic fractional chemotaxis model for tumor cell migration. *Mathematical Biosciences and Engineering*, *14*(1), 207-226.

Yang, X. J., Machado, J. T., &Baleanu, D. (2016). *Fractional Calculus with Applications in Mechanics: Vibrations and Diffusion Processes*. Wiley.

Zhang, H., Li, X., & Xu, Y. (2021). Solutions of fractional differential equations using the generalized Kudryashov method. *Applied Mathematical Modelling*, 95, 467-479.

Zhao, Y., Liu, C., & Wang, W. (2019). Application of the conformable fractional derivative to fluid mechanics. *Mathematical Problems in Engineering*, 2019.