DEFORMATION OF INTERNALLY PRESSURIZED HOLLOW CYLINDER OF A VULCANIZED RUBBER MATERIAL.

Abstract

The deformation of internally pressurized hollow cylinder made of vulcanized rubber material is considered. The analysis of the deformation resulted second order ordinary differential equation which sought for D operator Method of solution for the determination of displacement and stresses. Appropriate boundary conditions are set up in determining the constants involved in the solution. A closed solution for the displacement and stresses at any cross section of the cylinder was achieved.

Keywords: Vulcanized rubber, displacement, deformed radius, shear stresses and Hollow cylinder.

1. Introduction

The analysis of the deformation of internally pressurized hollow cylinder of a vulcanized rubber material is considered. The aim of the analysis is to determine the displacements and stresses caused by the internal pressure at any cross section of the hollow cylinder. The Vulcanized rubber which is a natural rubber mixed with sulphur during the processes of vulcanization. The sulphur added to the natural rubber increases the shear modulus of the rubber.Erumaka (1) worked on internally pressurized vulcanized rubber where he obtained a non trivial solution for the displacement.Ejike and Erumaka (2) worked on deformation of a rotating circular cylinder made of Blatz-ko material.Anani and Gholamhosein(3) worked on spherical material, Stress analysis of thick pressure vessel composed of incompressible hyperelastic materials where Neo Hookean strain energy function was used to determine the stress and displacement of spherical shell that is axisymmetric radially deformed under internal and external pressure. Exact solutions were derived for stress and stretch in a thick hyperelastic spherical shell and the effect of the structure parameter for different examples was discussed. Chung et al (4) analyzed the deformation of internally pressurized hollow cylinder and spheres for Blatz-ko type of compressible elastic material.The results showed that, when the ratio of the outer undeformed radius to the inner undeformed is higher than the critical value, the shear bifurcation occurs before the maximum pressure is reached, while the reverse occurs when the ratio is lower than the critical value. Huang [5] worked on finite displacement of a hollow sphere under internal and external pressures.Aani and Rahimi (6) investigated the displacement and stresses of axisymmetric radial deformation of the shell. They used Neo-Hookean strain energy function to obtain the behaviour of the material. The results presented show that the outer and inner radius is an important parameter which can be mirrored to some applications in order to control the stresses .Aani and Rahimi

(7). determined the stability of internally pressurized thick-walled spherical and cylindrical shells made of functionally graded incompressible. Kulcu (8) investigated a new strain energy function in other to describe the hyperelasticbehaviour of rubber-like materials under various deformation. The strain energy function represents an invariant based model which has two material constants. This model was tested with the experimental data of vulcanized rubbers, collagen and fibrin just like Levinson and Burgess did [14]. The parameters constants were kept constant when placed under certain types of loads. It was observed that there is an agreement between the model and the experimental data for all materials.Darijani and Bahremen (9) used polynomial hyperelastic models to obtain a closed form solution for analyses of rubbery solid circular cylinder. Robert et al (10) with the use of neo-Hookean and the Mooney–Rivlin models found the strain energy function for isotropic incompressible solids demonstrating a linear relationship between shear stress and amount of shear, and between torque and amount of twist, when subject to large simple shear or torsion deformations. Gao (11) in his work titledElasto-plastic analysis of an internally pressurized thick-walled cylinder using a strain gradient plasticity theory.The numerical data was demonstrated that the classical plasticity-based solution and the gradient plasticity-based solution predict almost identical results. Fracture mechanics analysis of cylindrical pressure vessels was carried out. Nabham et. al., (12) study the effect of the stress generated for an internally pressurized thick walled cylinders containing an internal radial hole using finite element method. His work shown that hoop stress increases due to increase of the hole parameter, diameter and depth. Moreover, the characterizations of notch may be used to determine the maximum stress limit.Darijani and Bahremen (13) used polynomial hyperelastic models to obtain a closed form solution for analyses of rubbery solid circular cylinder. Anani and Gholamhosein(14) worked on spherical material, Stress analysis of thick pressure vessel composed of incompressible hyperelastic materials where Neo Hookean strain energy function was used to determine the stress and displacement of spherical shell that is axisymmetric radially deformed under internal and external pressure. Elkholy et. al (15) study on Finite Element Analysis of Stresses Caused by External Holes in Hydraulic Cylinders.In this present paper we sought for a closed solution using D-operator method of solution of second order ordinary differential equation for the determination of stresses and displacement across a hollow cylindrical pipe made of vulcanized rubber material.

2. Formulation of the deformation equation

Let consider an open region$D_0 = \{(R, \Theta , Z): a < R < b, 0 < \Theta < 2\pi\}$ denote the cross section of a right circular pipe with inner radius a and outer radius b in its undeformed configuration. The pipe is subjected to a uniform internal pressure of magnitude ρ. The resulting deformation is a one to one axisymmetric deformation which maps the point with cylindrical polar coordinate $(R, \Theta\ , Z)$ in the undeformed configurationD_0 to the point $(R, \Theta\ , Z)$ in the deformed region D such that

$r = r(R),\ \theta = \Theta,\ z = Z$(1)

where $r(R) \in C^2[a, b]$ is to be obtained

The deformation gradient tensor F for equation (1) is given as

$$\bar{F} = \begin{pmatrix} \frac{\partial r}{\partial R} & \frac{1}{R}\frac{\partial r}{\partial R} & \frac{\partial r}{\partial Z} \\ r\frac{\partial \theta}{\partial R} & \frac{r}{R}\frac{\partial \theta}{\partial \Theta} & r\frac{\partial \theta}{\partial Z} \\ \frac{\partial z}{\partial R} & \frac{1}{R}\frac{\partial r}{\partial R} & \frac{\partial z}{\partial Z} \end{pmatrix} = \begin{pmatrix} r_R & 0 & 0 \\ 0 & \frac{r}{R} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2)$$

The Left Cauchy-Green deformation gradient tensor B associated with (1) is given as

$$\bar{B} = \bar{F}\bar{F}^T = \begin{pmatrix} r_R & 0 & 0 \\ 0 & \frac{r}{R} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} r_R & 0 & 0 \\ 0 & \frac{r}{R} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} (r_R)^2 & 0 & 0 \\ 0 & (\frac{r}{R})^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} (3)$$

$$\bar{B}^2 = \bar{B}.\bar{B} = \begin{pmatrix} (r_R)^2 & 0 & 0 \\ 0 & (\frac{r}{R})^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} (r_R)^2 & 0 & 0 \\ 0 & (\frac{r}{R})^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} (r_R)^4 & 0 & 0 \\ 0 & (\frac{r}{R})^4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The three principal strain invariants I_1 , I_2 , and I_3 given as

I_1 = trace $\bar{B}$, $I_2 = \frac{1}{2}[(tr\bar{B})^2 - tr\bar{B}^2]$, I_3= det $\bar{B}$

$$I_2 = \frac{1}{2}[(tr\bar{B})^2 - tr\bar{B}^2] = \frac{1}{2}\left[\left((r_R)^2 + \left(\frac{r}{R}\right)^2 + 1\right)^2 - ((r_R)^4 + \left(\frac{r}{R}\right)^4 + 1)\right]$$

$$I_1 = tr\bar{B} = (r_R)^2 + \left(\frac{r}{R}\right)^2 + 1,\ I_2 = (\frac{rr_R}{R})^2 + (r_R)^2 + \left(\frac{r}{R}\right)^2,\ I_3 = det\bar{B} = (\frac{rr_R}{R})^2 \quad (4)$$

Where $W(I_1, I_2, I_3)$ is the strain energy function and $W_i = \frac{\partial W}{\partial I_i}$, $i = 1, 2, 3$.

Here we consider compressible isotropic elastic vulcanized rubber material characterized by the elastic potential

$W(I_1, I_3) = \frac{\mu}{2}\left[I_1 - 3I_3^{\frac{1}{2}}\right] + \frac{1}{2}I_3^{\frac{1}{2}}$, where μ>0 (a constant) is the shear modulus for infinitesimal deformation.

$$\frac{\partial W}{\partial I_1} = \frac{\mu}{2}, \qquad \frac{\partial W}{\partial I_2} = 0, \frac{\partial W}{\partial I_3} = I_3^{\frac{-1}{2}}\left[\frac{1-3\mu}{4}\right] = k_1 I_3^{\frac{-1}{2}}. \quad \text{where} k_1 = \frac{1-3\mu}{4}$$

3. **Stress tensor $\boldsymbol{\tau}$:** The stress tensor for compressible vulcanized rubber material is given by

$\tau = \varphi_0 I + \varphi_1 \bar{B} + \varphi_{-1}\bar{B}^{-1}$(5)

$$\varphi_0 = 2I_3^{\frac{-1}{2}}[I_2 W_2 + I_3 W_3] = 2I_3^{\frac{1}{2}} W_3 = 2k_1 = \frac{1-3\mu}{2} = k_2$$

$$\varphi_1 = 2I_3^{\frac{-1}{2}} W_1 = \frac{\mu R}{r_R r},\quad \varphi_{-1} = -2I_3^{\frac{1}{2}} W_2 = 0$$

Equation (5) reduces to

$\tau = \varphi_0 I + \varphi_1 \bar{B}$

$$\tau = k_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{\mu R}{r_R r} \begin{pmatrix} (r_R)^2 & 0 & 0 \\ 0 & (\frac{r}{R})^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} (6)$$

In components form of the polar cylindrical material

$$\tau = \begin{pmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \tau_{zz} \end{pmatrix} \quad (7)$$

Comparing (6) and (7), we have

$$\tau_{rr} = k_2 + \frac{\mu R r_R}{r} (8.1)$$

$$\tau_{\theta\theta} = k_2 + \frac{\mu R}{r_R r} (8.2)$$

$$\tau_{zz} = k_2 + \frac{\mu R}{r_R r} (8.3)$$

$$\tau_{\theta r} = \tau_{r\theta} = \tau_{zr} = \tau_{rz} = \tau_{z\theta} = \tau_{\theta z} = 0 \ (8.4)$$

where $k_2 = \frac{1-3\mu}{2}$ and $a < r < b$

4. Equation of Equilibrium

The equilibrium equation is given by

$$\text{Div}\overline{T} = 0 \quad (9)$$

In component form, we have

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{1}{r}(\tau_{rr} - \tau_{\theta\theta}) + B_r = 0 \qquad (10.1)$$

$$\frac{\partial \tau_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2}{r}\tau_{r\theta} + B_\theta = 0 \qquad (10.2)$$

$$\frac{\partial \tau_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{z\theta}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} + \frac{1}{r}\tau_{zr} + B_z = 0 \qquad (10.3)$$

where B_r, B_θ *and* B_z are components of the body forces in r, θ and z vectors respectively. In the absence of body force and the non-zero component of equilibrium equation is the radial component given by

$$\frac{d\tau_{rr}}{dr} + \frac{1}{r}\frac{d\sigma_{r\theta}}{d\theta} + \frac{d\tau_{rz}}{dz} + \frac{1}{r}(\tau_{rr} - \tau_{\theta\theta})=0 \qquad (11)$$

Since $\frac{d\tau_{r\theta}}{d\theta} = 0$, $\frac{d\tau_{r\theta}}{d\theta} = 0$ then (11) becomes

$$\frac{d\tau_{rr}}{dr} = \frac{1}{r}(\tau_{\theta\theta} - \tau_{rr})$$

$$\frac{\partial \tau_{rr}}{\partial R}\frac{\partial R}{\partial r} + \frac{1}{r}(\tau_{rr} - \tau_{\theta\theta}) = 0 \qquad (12)$$

Substituting (8) in (12) we have

$$\frac{\partial}{\partial R}\left(\frac{\mu R r_R}{r}\right)\frac{\partial R}{\partial r} + \frac{1}{r}\left(\frac{\mu R r_R}{r} - \frac{\mu R}{r_R R}\right) = 0$$

$$\left(\frac{\mu r_R}{r} - \frac{\mu (r_R)^2 R}{r^2} + \frac{R\mu r_{RR}}{r}\right)\left(\frac{1}{r s_R}\right) + \frac{1}{r}\left(\frac{\mu R r_R}{r} - \frac{\mu R}{r_R R}\right) = 0$$

$$\mu(R r_R + R^2 r_{RR} - r) = 0$$

$\mu \neq 0$, $\quad R r_R + R^2 r_{RR} - r = 0$

Therefore $\quad R^2 r_{RR} + R r_R - r = 0$(13)

Let $R = e^z$, $z = logR$

$$\frac{dr}{dR} = \frac{dr}{dz}\frac{dz}{dR} = \frac{1}{R}\frac{dr}{dz}$$

$R\frac{dr}{dR} = \frac{dr}{dZ}$(14)

$$\frac{d^2 r}{dR^2} = \frac{1}{R^2}\left(\frac{d^2 r}{dz^2} - \frac{dr}{dz}\right)$$

Therefore $R^2 \frac{d^2r}{dR^2} = \left(\frac{d^2r}{dz^2} - \frac{dr}{dz}\right)$ (15)

Using (14) and (15), (13) becomes

$$\left(\frac{d^2r}{dz^2} - \frac{dr}{dz}\right) + \frac{dr}{dz} - r = 0$$

$$\frac{d^2r}{dz^2} - r = 0$$

Let $\frac{d}{dz} = D$

Then (7) becomes

$(D^2 - 1)r = 0$(16)

The auxillary equation of equation (16) is given as

$m^2 - 1 = 0$

$m = \pm 1$

$r(R) = A_1e^z + A_2e^{-z}$

$r(R) = A_1e^{logR} + A_2e^{-logR}$

$r(R) = A_1R + \frac{A_2}{R}$(17)

5. Boundary conditions

Since the pressure act internally we can set up a boundary conditions of the form

$r(a) = -\varphi \text{ and } r(b) = 0$(18)

Using equation (18) in equation (17), we have equations (19) and (20) given below as

$\varphi = A_1a + \frac{A_2}{a}$(19)

$0 = A_1b + \frac{A_2}{b}$(20)

Solving (19) and (20) simultaneously we obtain

$$A_1 = \frac{a\varphi}{b^2 - a^2}, \quad and A_2 = \frac{-a\varphi b^2}{b^2 - a^2}$$

Therefore the displacement $r(R)$ is given as

$$r(R) = \frac{a\varphi R}{b^2 - a^2} - \frac{a\varphi b^2}{(b^2 - a^2)R} \quad (21)$$

$$\dot{r} = \frac{a\varphi}{b^2 - a^2} + \frac{a\varphi b^2}{(b^2 - a^2)R^2} \quad (22)$$

Using equation (21) and (22) in (8) we obtain the components of the stresses.

6. Result and discussion

Let $\varphi = 0.75mm$, $a = 10mm$, $b = 30mm$, $\mu = 0.0006Gpa$ in (23) and (8.1) to obtain Table 1.

Table 1: The table shows values of the radius in mm and the corresponding values of the deformed radius and displacement gradient.

R	r(R)	$\dot{r}$	$\tau_{rr} = -\rho$
10	-0.7500	0.09375	-0.49910
12	-0.5906	0.06797	-0.49910
14	-0.4714	0.05242	-0.49910
16	-0.3773	0.04233	-0.49910
18	-0.3000	0.03517	-0.49910
20	-0.2344	0.03047	-0.49910
22	-0.1773	0.02681	-0.49910
24	-0.1266	0.02402	-0.49910
26	-0.0808	0.02186	-0.49910

28	-0.0388	0.02014	-0.49910
30	-0.0000	0.01875	-0.49910

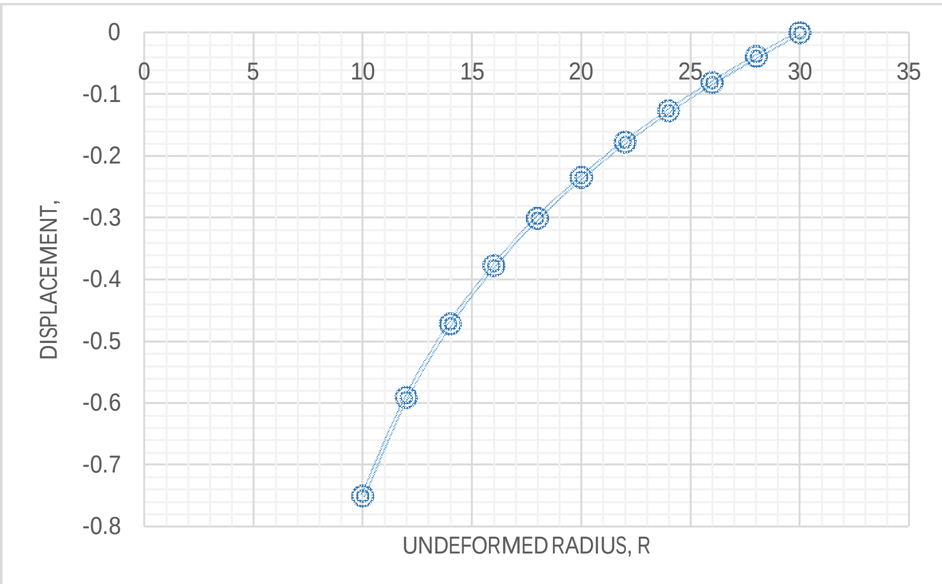

Figure 1: A graph of equation (21) with undeformed radius plotted against displacement.

As the radius of the hollow cylinder made of vulcanized rubber increases the displacement of the material increases.

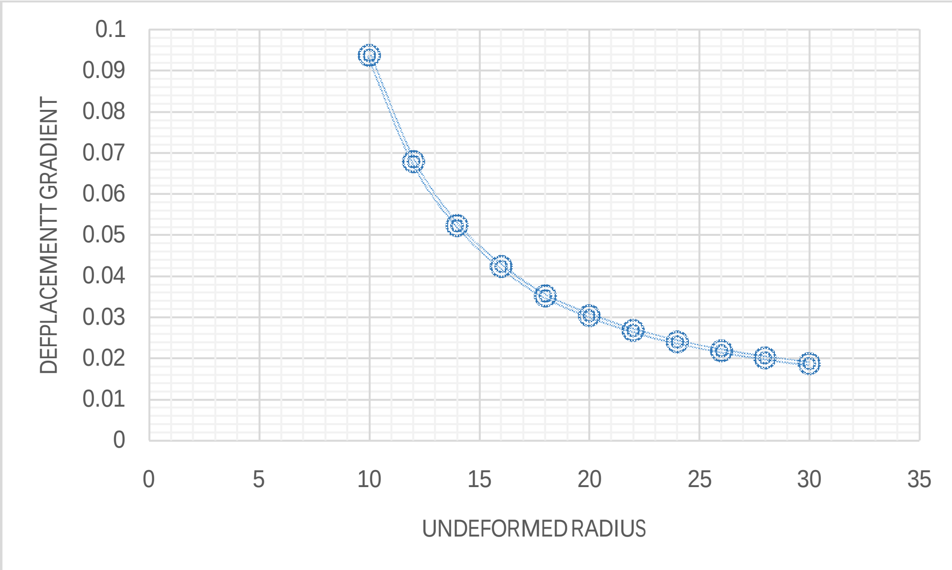

Figure 2: A graph of equation (22) with undeformed radius plotted against deplacement gradient.

As the radius of the hollow cylinder made of vulcanized rubber increases the deplacement gradient decreases.

7.Conclusion

This present work establish a closed solution for the displacement and stresses as resulting of internally pressurized hollow cylindrical pipe made of vulcanized rubber material. A graph of undeformed radius against displacement is plotted as shown in figure 1. Equation (21) gives the displacement. The components of the stress at any cross section of the hollow cylindrical pipe made of vulcanized rubber material was obtained. It was observed that as the radius of the hollow cylinder made of vulcanized rubber increases the deformation gradient decreases and the radius of the hollow cylinder made of vulcanized rubber increases with increase in the displacement of the material.

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